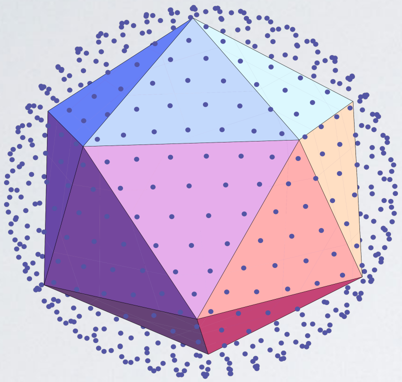
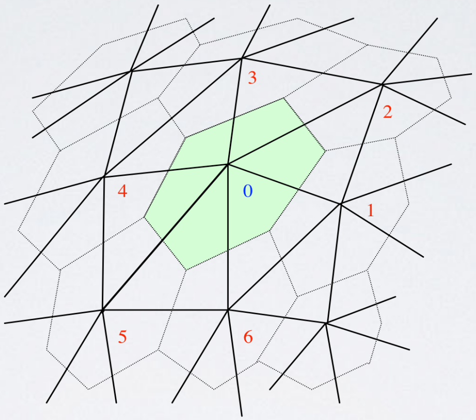
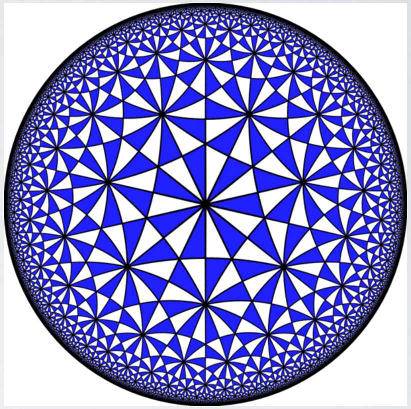
LATTICE QUANTUM FINITE ELEMENTS (QFE) FOR CONFORMAL FIELD THEORY







Rich Brower, Boston University

RBRC Workshop on Lattice Gauge Theories 2016 — March 10

See **Quantum Finite Elements for Lattice Field Theory** M. Cheng, G. Fleming, A. Gasbarro, T. Raben, C-I Tan & E. Weinberg Latt15 http://arxiv.org/abs/1601.01367

Our Quantum Finite Element (QFE) method is at the intersection of two traditions

I. CLASSICAL FEM# for PDEs on *smooth Riemann Manifolds*

FEM: Alexander Hrennikoff (1941) Richard Courant (1943)*

Discrete Exterior Calculus* (de Rahm Complex, Whitney, etc, etc.),

II. QUANTUM FEILDS on *random Lattices.*

Regge Calculus T. Regge, Nuovo Cimento 19 (1961) 558. *

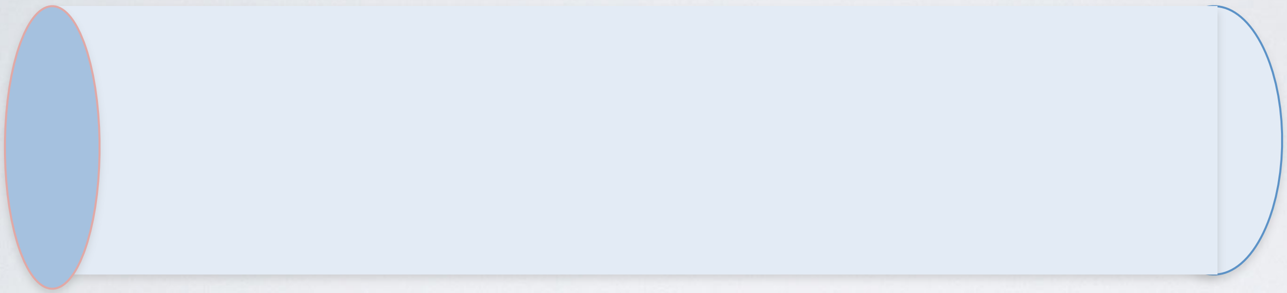
Random Lattices: N. H. Christ, R. Friedberg, and T. D. Lee, Nucl. Phys. B 202, 89 (1982).* Fermion Fields on a Random Lattice: R. Friedberg, T.D.Lee and Hai-Cang Ren Prog. of Th. Physics 86 (1986).

Topology/Chirality 'tHooft, Leuscher et al for QCD

 # Google: "Finite Element Method" ==> 25,500,000 results (0.46 seconds)

MOTIVATION*
RADIAL QUANTIZATION OF CONFORMAL FIELD THEORY

$\mathbb{R} \times \mathbb{T}^3$ vs $\mathbb{R} \times \mathbb{S}^3$ ← Boundary of Global AdS5



← Isometries are the conformal group O(5,1)

On lattice scales exponentially

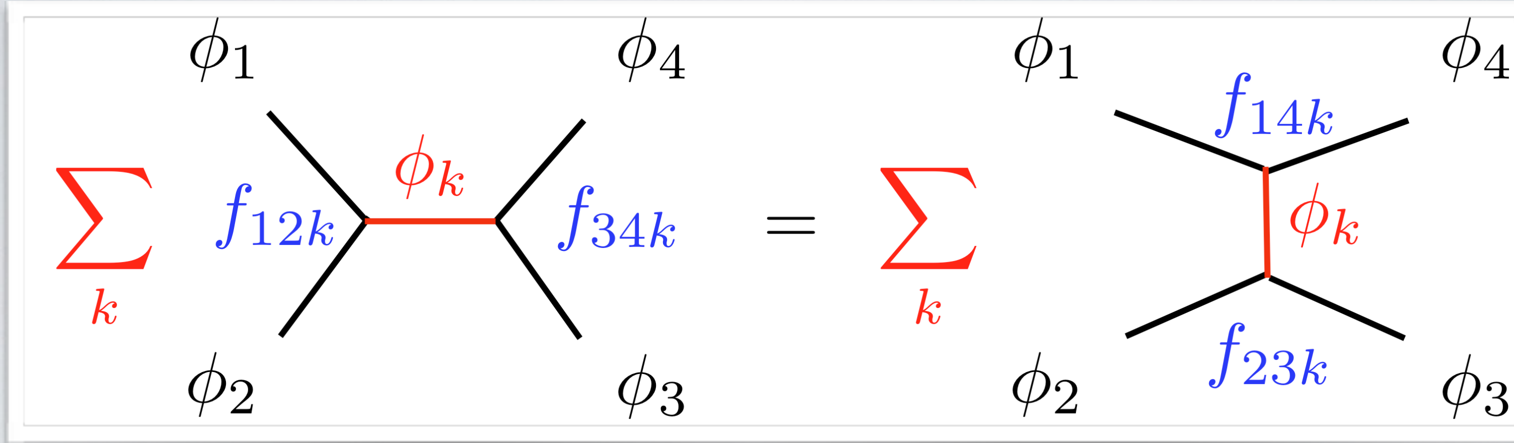
$$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

$$1 < t < aL \implies 1 < \tau = log(r) < L$$

* See **"Lattice Radial Quantization: 3D Ising"** by Brower, Fleming and Neuberger Phys.Lett. B721 (2013) 299-305.

Potential Application: (1) BSM composite Higgs Model Building (2) AdS/CFT weak-strong duality, (3) Critical Phenomena in general, etc

CFT OPE expansion and Conformal Bootstap



$$\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$$

Only "tree" diagrams!
"partial waves" exp: sum over conformal blocks

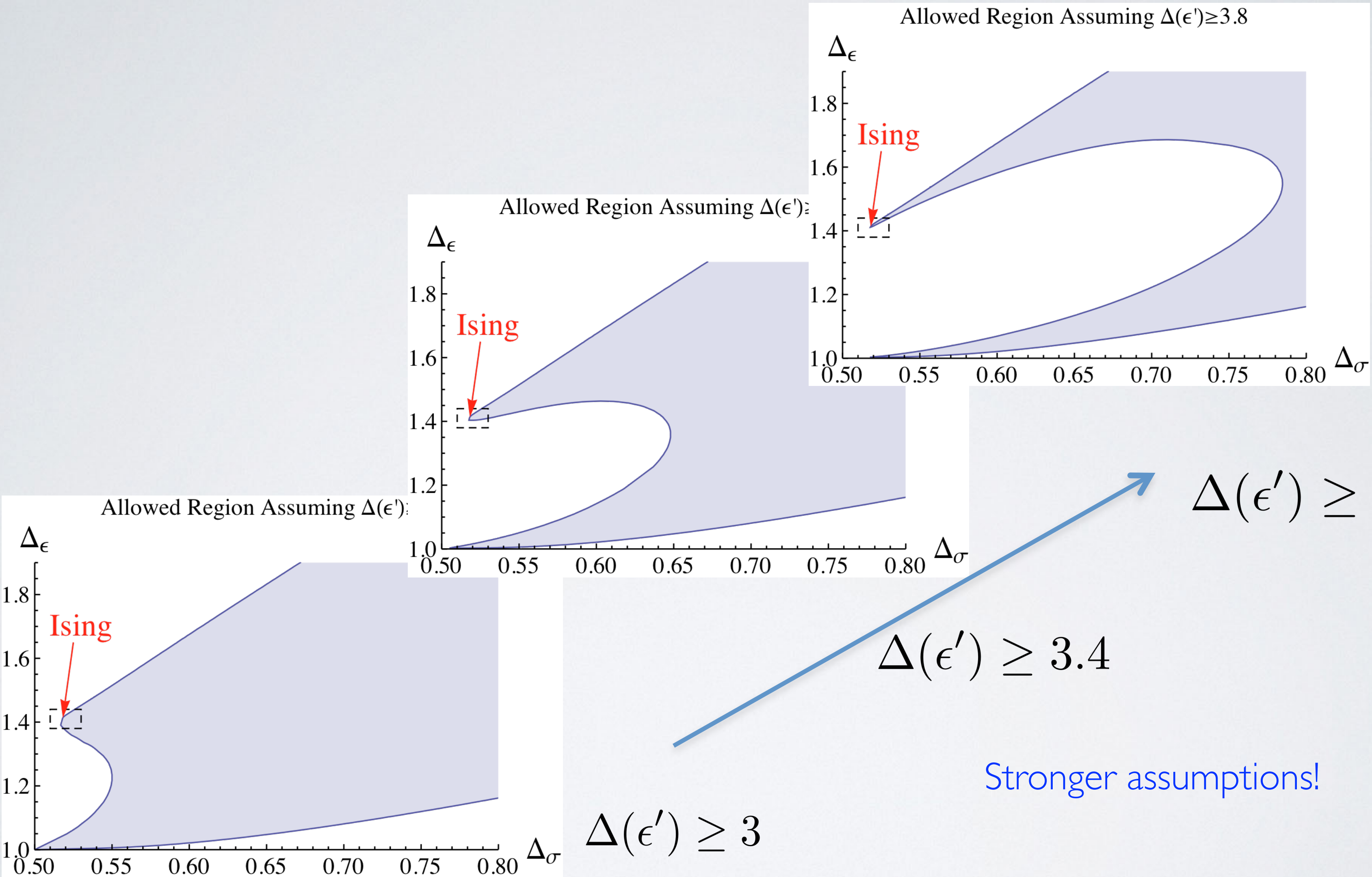
$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(0)$$

Exact 2 and 3 correlators

(i.e. Data: spectra + couplings to conformal blocks)

CFT Bootstrap: OPE & factorization completely fixed the theory

/NEQUALITIES FROM BOOTSTRAP*



$\Delta(\epsilon') \geq 3.8$

$\Delta(\epsilon') \geq 3.4$

$\Delta(\epsilon') \geq 3$

Stronger assumptions!

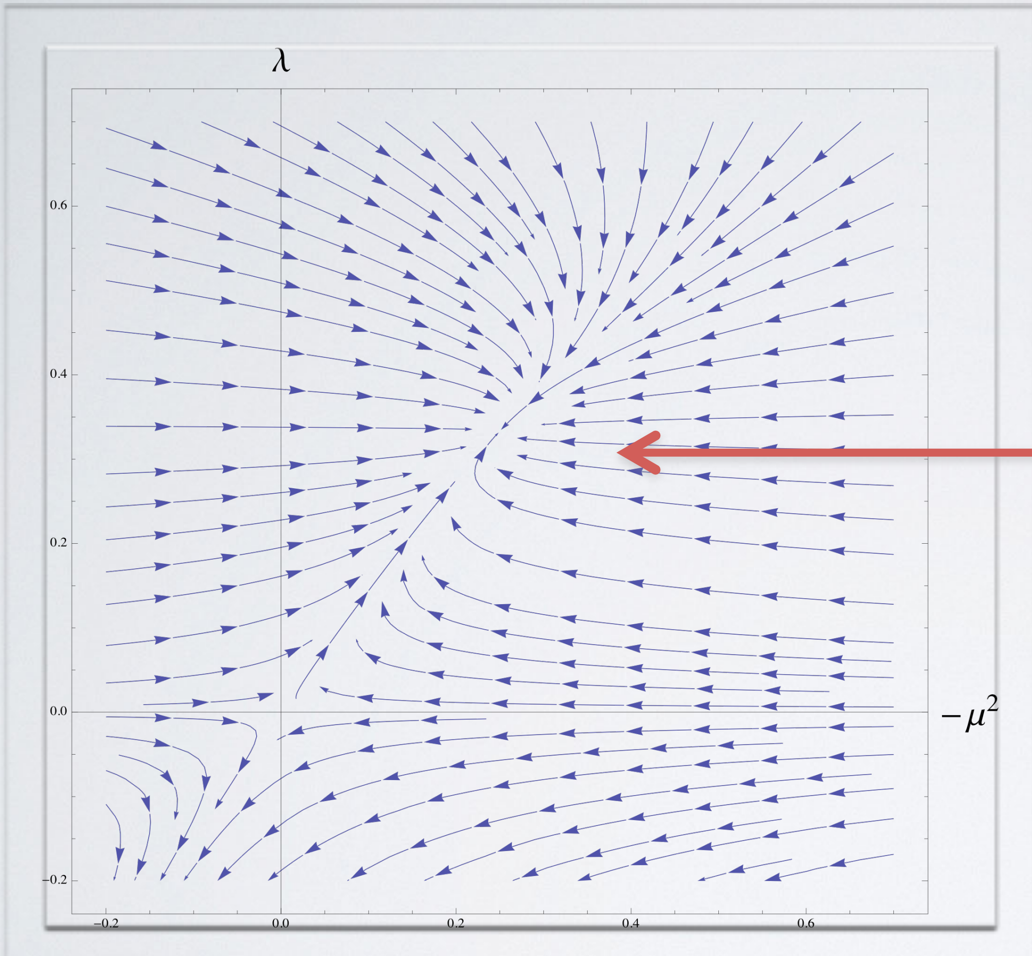
- *“**Solving the 3D Ising Model with the Conformal Bootstrap**” (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

PLACING QUANTUM FIELDS ON A RIEMANN SIMPLICIAL COMPLEX

- Scalar Theory: Classical FEM Limit
- Dirac Theory: Lattice Spin Connection
- Renormalization: 2D Test of Counter Terms
- Future: D > 2 & Gauge Theories

SCALAR FIELD

Replace Ising Model by phi 4th



Wilson-Fisher FP

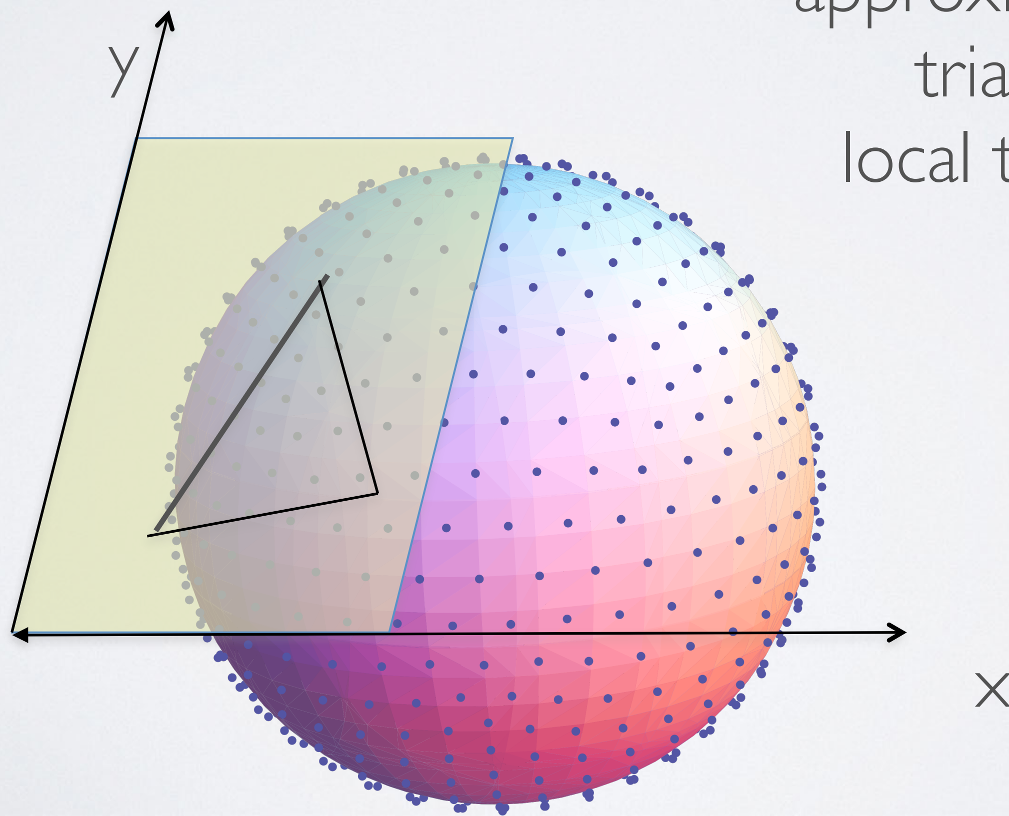
Gaussian FP

$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu^2/2\lambda)^2$$

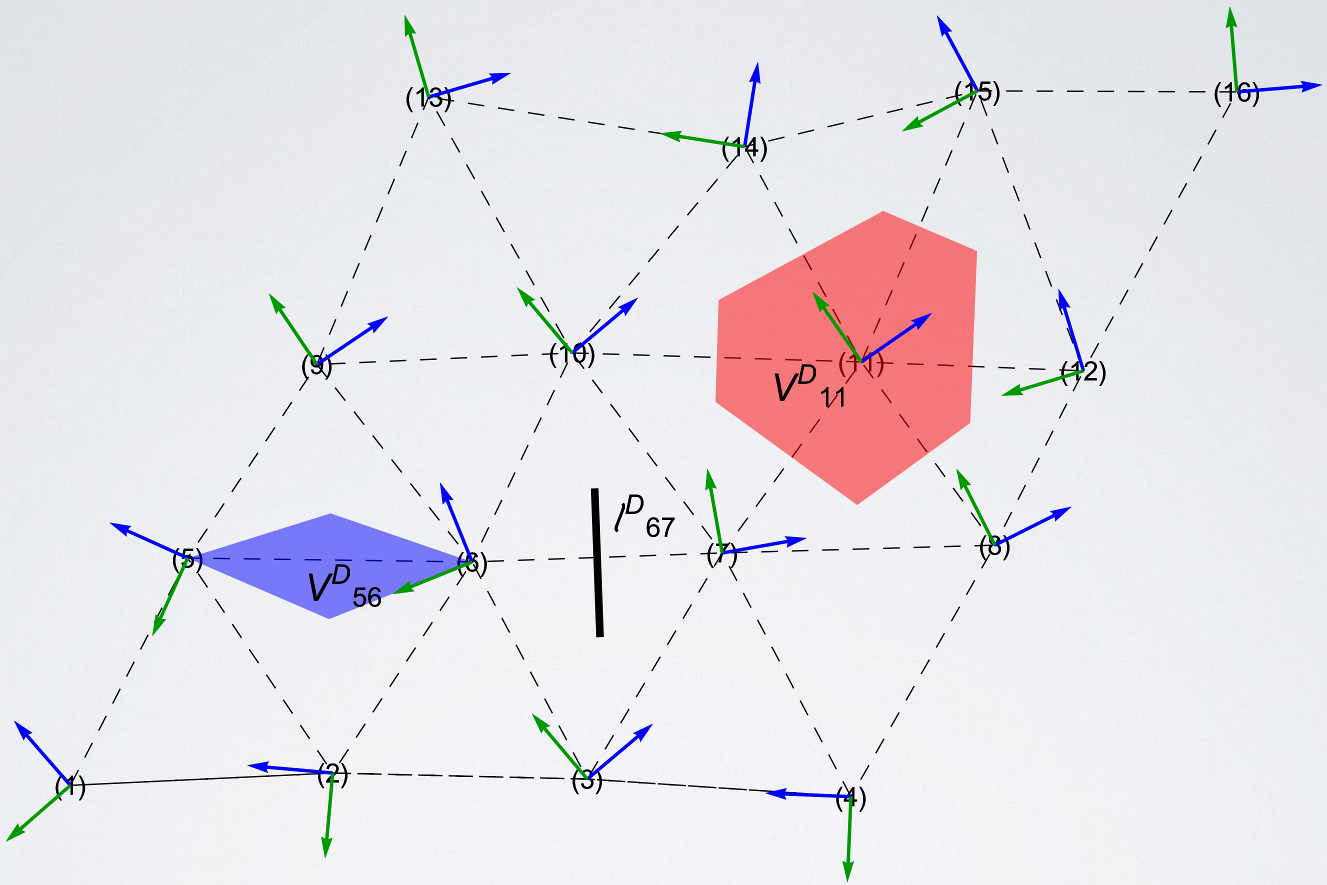
TEST CFT: PHI 4TH WILSON-FISHER FIXED POINT IN 3D.

$$L = \int d^3x[\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda\sqrt{g}(\phi^2 - \mu^2/2\lambda)^2]$$

approximate spherical triangles onto local tangent plane



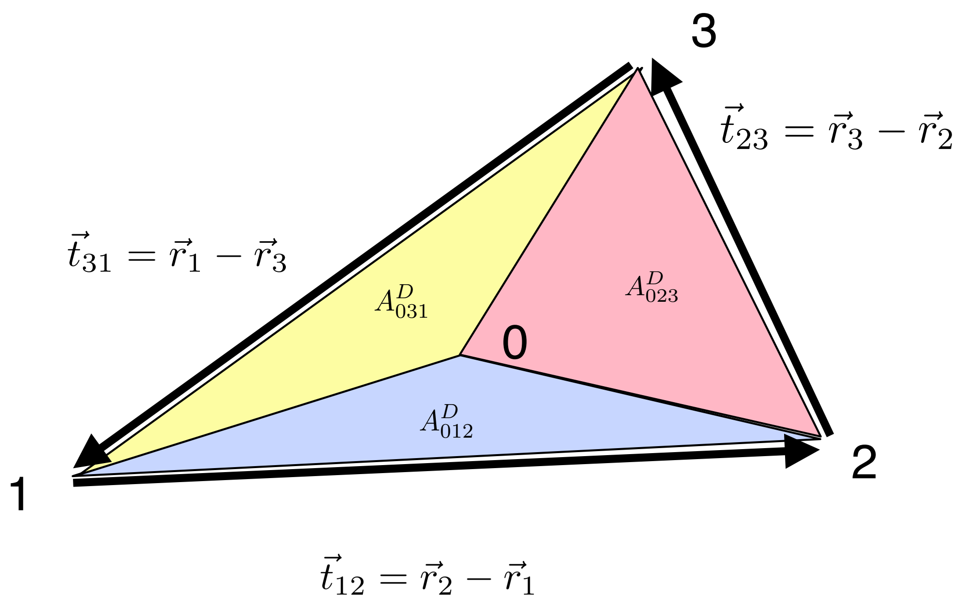
Simplicial Complex: Lattice vs dual Lattice. Discrete Exterior Calculus

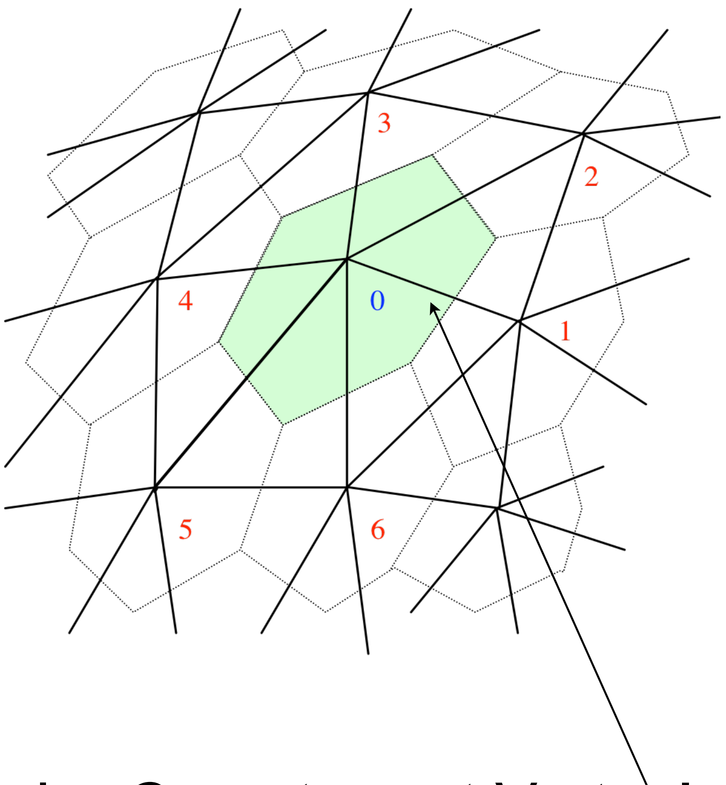


1. First replace the smooth Riemann manifold $(\mathcal{M}, g)$ by an approximating piecewise flat manifold $(\mathcal{M}_\sigma, g_\sigma)$ composed of elementary simplices.

2. Second expand the field, $\phi(x)$, in a finite element basis on each simplex: $\phi(x) \simeq W^i(x)\phi_i$.

FEM geometry on edges.

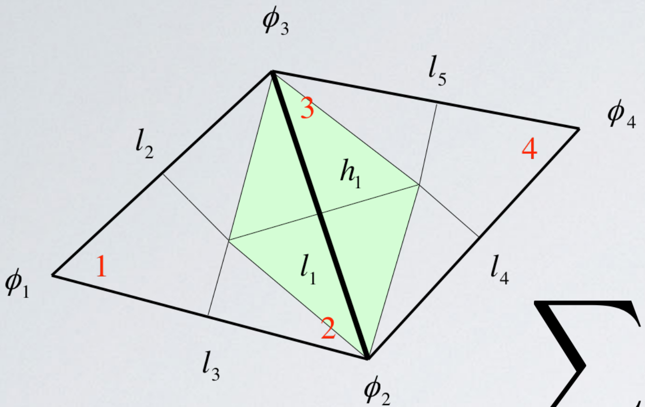




Singular Curvature at Vertex!

The l's fix metric and the local co-ordinates (diffeomorphism) and the angles the intrinsic curvature.

REGGE CALCULUS FORMULATION



LINEAR FEM/ REGGE CALCULUS *

$$\sum_{\triangle_{kij}} \sqrt{g(k)}g^{ij}(k)\frac{(\phi_k-\phi_i)(\phi_k-\phi_j)}{l_{ki}l_{kj}}$$

Only for D = 2

$$\text{Delaunay Link Area:} \quad A_d = h_1 l_1$$

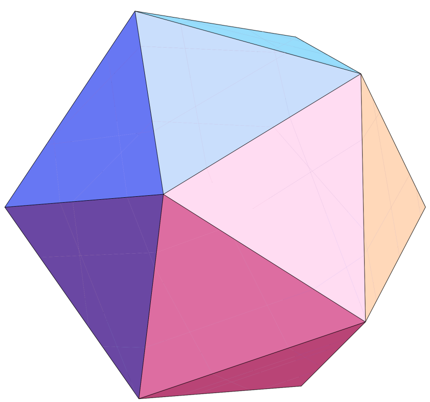
$$FEM : A_d \frac{(\phi_2-\phi_3)^2}{l_1^2}$$

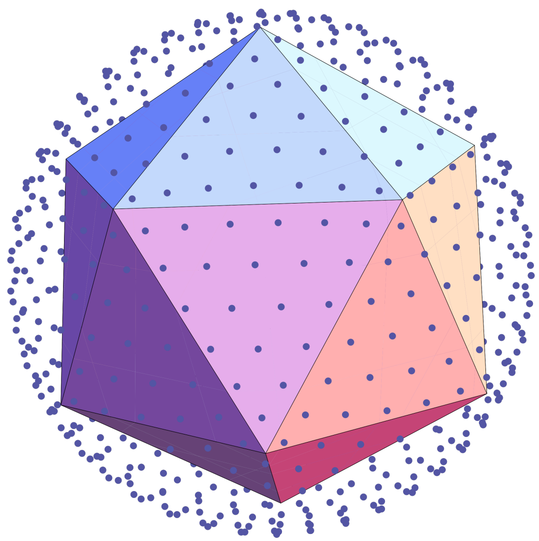
DISCRETE EXTERIOR CALCULUS
or
CHRIST FRIEBERG & LEE

* H. Hamber, S. Liu, **Feynman rules for simplicial gravity,** NP B475 (1996)

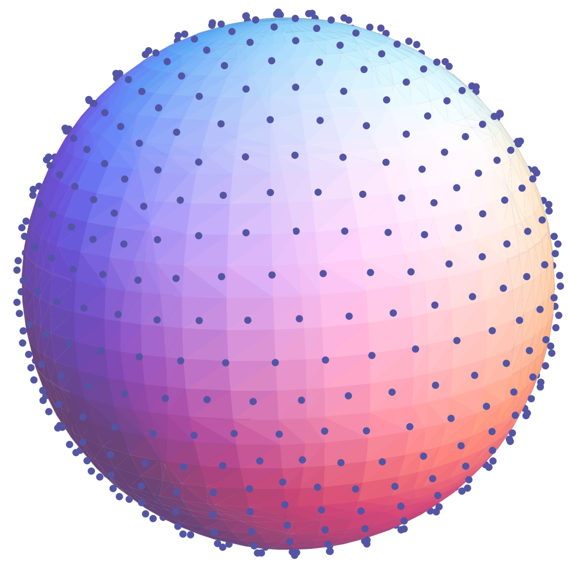
Order s Refined Triangulated Icosahedron

$s = 1$





$s = 8$

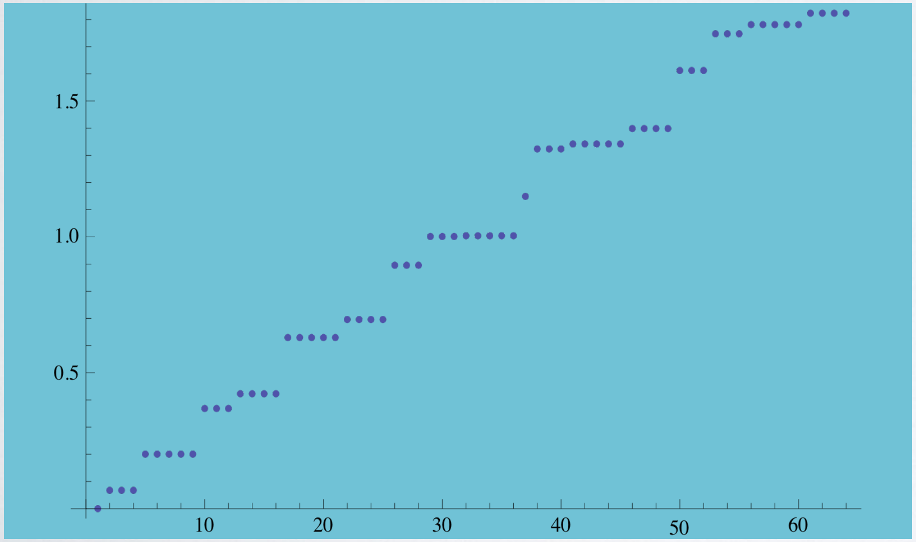


I = 0 (A),1 (T1) , 2 (H) are irreducible 120
Iscosahedral subgroup of O(3)

FEM FIXES THE HUGE SPECTRAL DEFECTS OF THE LAPLACIAN ON THE SPHERE

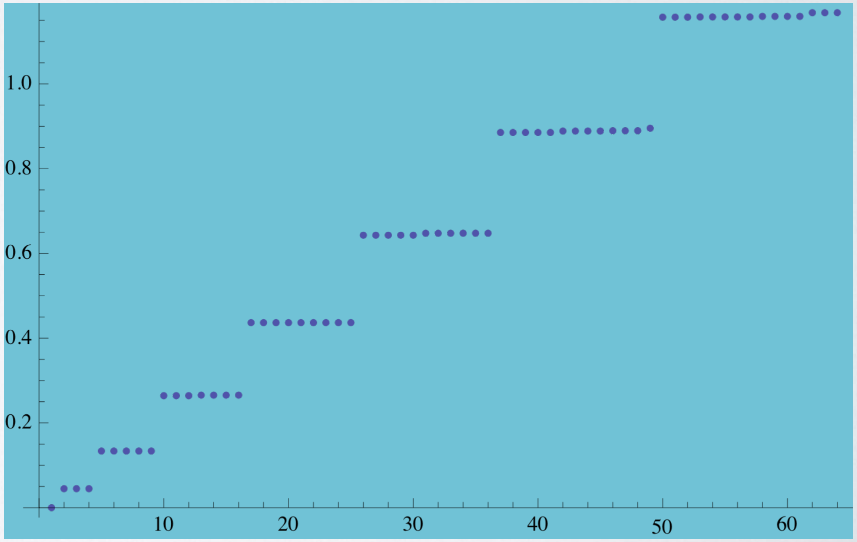
For s = 8 first (l+1)*(l +1) = 64 eigenvalues

BEFORE (K = 1)



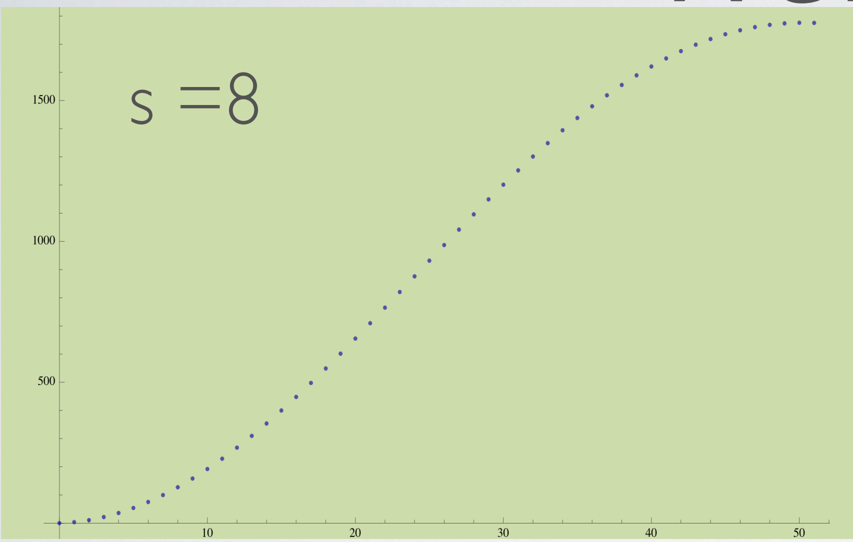
l, m

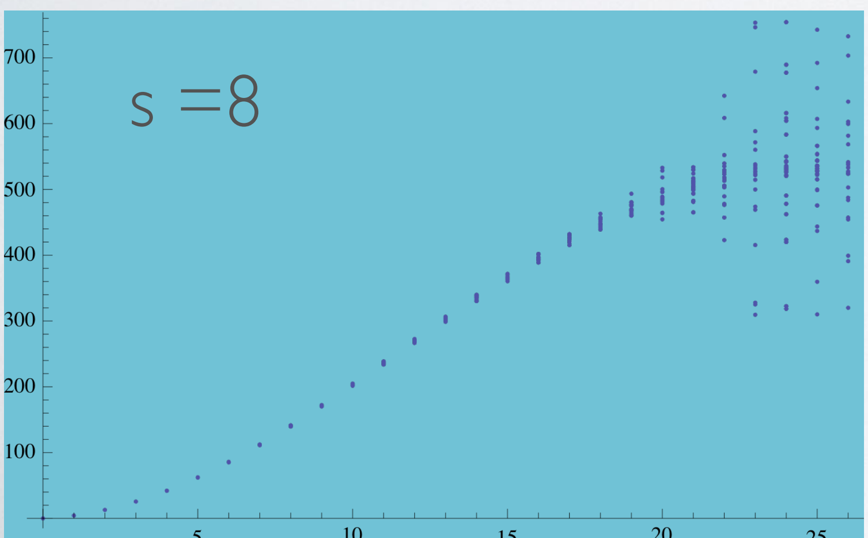
AFTER (FEM K's)

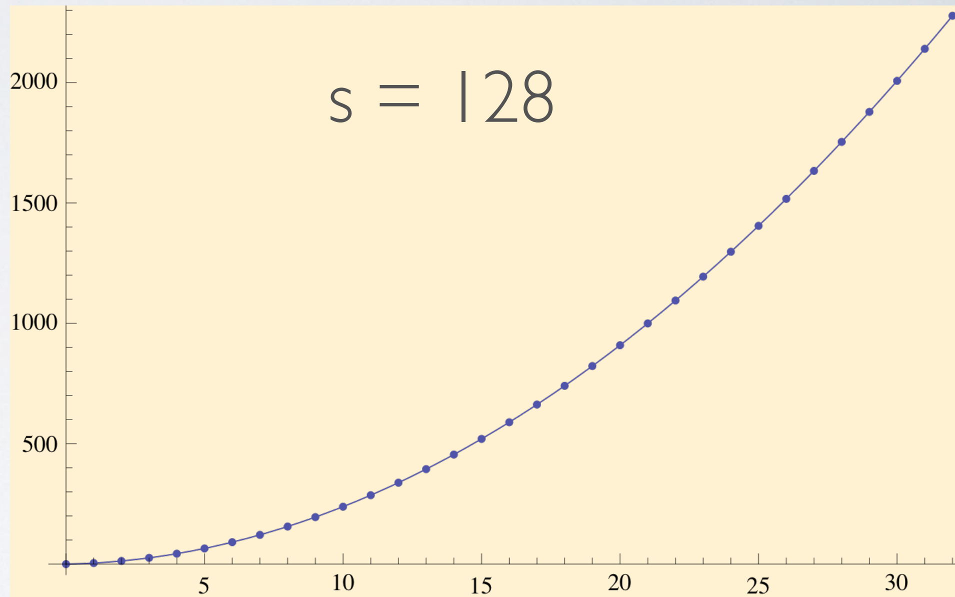


l, m

SPECTRUM OF FE LAPLACIAN ON A SPHERE







Fit

$$l + 1.00012\, l^2$$
$$- 13.428110^{-6}\, l^3 - 5.5724410^{-6}\, l^4$$

DIRAC FIELD ON REIMANN MANIFOLD

QFEM DIRAC EQUATION: MUCH HARDER

$$S = \frac{1}{2}\int d^D x \sqrt{g}\bar{\psi}[\mathbf{e}^\mu(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$$\mathbf{e}^\mu(x) \equiv e^\mu_a(x)\gamma^a$$ Verbein & Spin connection*

$$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_\mu(x)\sigma_{ab} \quad , \quad \sigma_{ab} = i[\gamma_a, \gamma_a]/2$$

(1) New spin structure "knows" about intrinsic geometry

(2) Need to avoid simplex curvature singularities at sites.

(3) Spinors rotations (Lorentz group) is double of O(D).

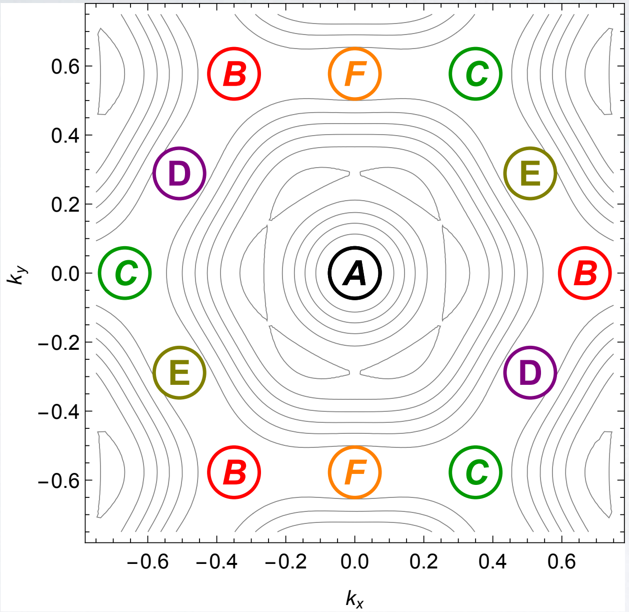
$$e^{i(\theta/2)\sigma_3/2} \to -1 \qquad \text{as} \quad \theta \to 2\pi$$

---

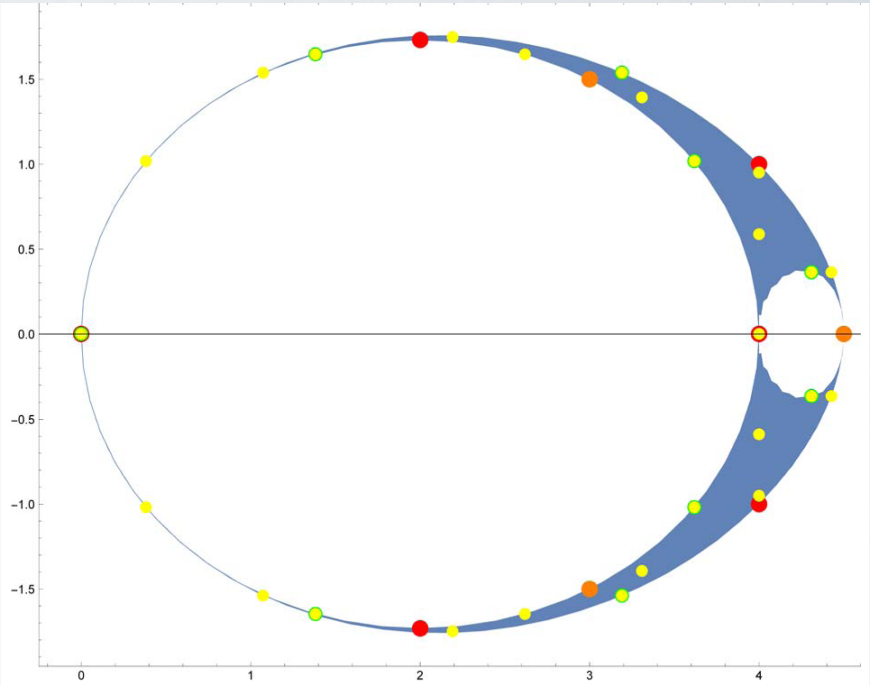
* Must satisfy the tetrad postulate!

$$\omega^{ab}_\mu = \frac{1}{2}e^{\nu[a}(e^{b]}_{\nu,\mu} - e^{b]}_{\mu,\nu} + e^{b]\sigma}e^c_\mu e_{\nu c,\sigma})$$

2D DIRAC TRIAGULAR LATTICE



Torus: NAIVE



Torus: With Wilson Term

9 pts (orange) 16 pts (red) 25 pts(green) 100 pts (yellow)

Continuum Acton

$$S = \int d^D x \sqrt{g}\, \bar{\psi}[\mathbf{e}^{\mu}(\partial_\mu - i\boldsymbol{\omega}_\mu(x)) + m]\psi(x) \;,$$

Tetrad Postulate

$$\partial_\mu \mathbf{e}^{\nu} + \Gamma^{\nu}_{\mu,\lambda}\mathbf{e}^{\lambda} = i[\boldsymbol{\omega}_\mu, \mathbf{e}^{\nu}] \;.$$

$$\boxed{\boldsymbol{D}_\mu}$$

Simplicial Lattice Action

$$S_\sigma = \frac{1}{2}\sum_{\langle ij \rangle} \frac{V^D_{ij}}{l_{ij}} (\bar{\psi}_i e^{(i)j}_a \gamma^a \Omega_{ij}\psi_j + \bar{\psi}_j e^{(j)i}_a \gamma^a \Omega_{ji}\psi_i) + \cdots$$

$$\psi_i \to \Lambda_i \psi \quad , \quad \bar{\psi}_j \to \bar{\psi}_j \Lambda^\dagger_j \quad , \quad \mathbf{e}^{(i)j} \to \Lambda_i \mathbf{e}^{(i)j}\Lambda^\dagger_i \quad , \quad \Omega_{ij} \to \Lambda_i \Omega_{ij} \Lambda^\dagger_j$$

WILSON/CLOVER TERM

$$[\gamma_\mu(\partial_\mu - iA_\mu)]^2 = (\partial_\mu - iA_\mu)^2 - \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} \ ,$$

$$[\mathbf{e}^\mu_a(\partial_\mu - i\boldsymbol{\omega}_\mu)]^2 = \frac{1}{\sqrt{g}}\boldsymbol{D}_\mu\sqrt{g}g^{\mu\nu}\boldsymbol{D}_\nu - \frac{1}{2}\sigma^{ab}e^\mu_a e^\nu_b \boldsymbol{R}_{\mu\nu}$$

$$S_{Wilson} = \frac{r}{2}\sum_{\langle i,j\rangle}\frac{aV_{ij}}{l_{ij}^2}(\bar{\psi}_i - \bar{\psi}_j\Omega_{ji})(\psi_i - \Omega_{ij}\psi_j)$$

Construction Procedure for Discrete Spin connection

(1) Assume Elements with Spherical Triangles (i,j,k) or boundaries give by geodesics on an 2D manifold

(Angles at each vertex add to 2 pi exactly)

(2) Calculate discrete "curl" around the triangle

$$\Omega_{ij}\Omega_{jk}\Omega_{ki} = e^{i(2\pi - \delta_\triangle)\sigma_3/2}$$

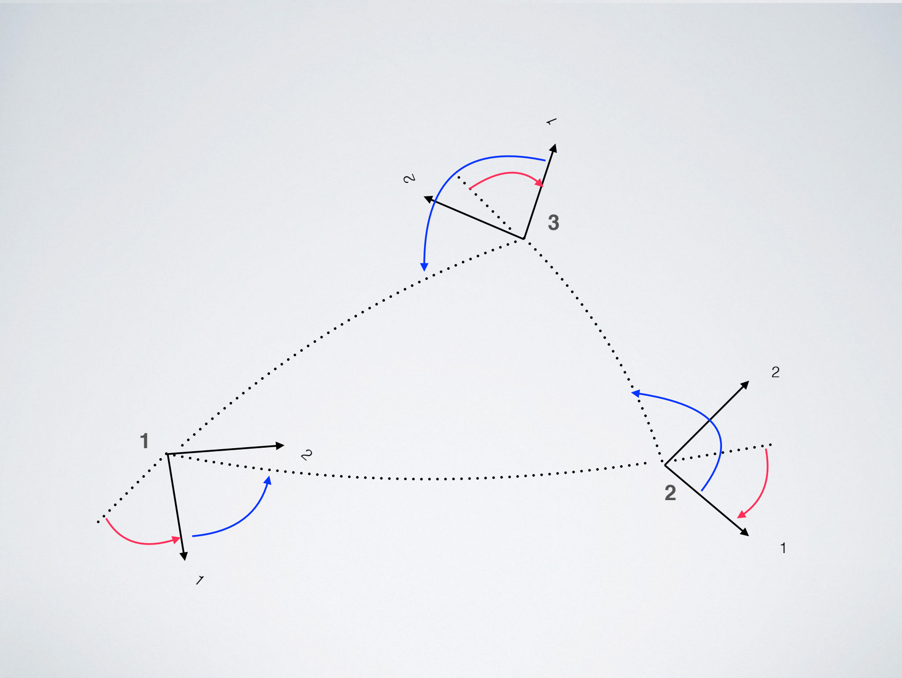
(3) Fix $\Omega_{ij} \to \pm\Omega_{ij}$ so $\delta_\triangle \sim A_{ijk}/4\pi R$

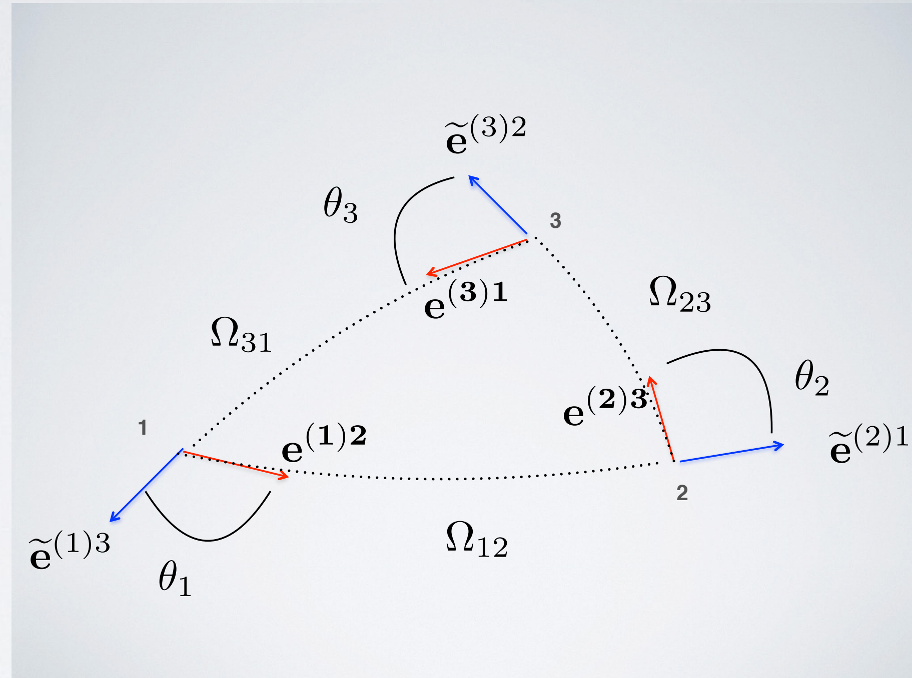
Sphere: or any manifold with this topology has a unique lattice spin connection up to gauge Lorentz transformation on spinors

Torus: There are 4 solutions: (periodic/anti-periodic): Non-contractible loops.

Category Theory: A spin structure is a property shared between any simplicial complex and Riemann manifolds to which they correspond.

Lattice Spin Connection on simplicial lattice



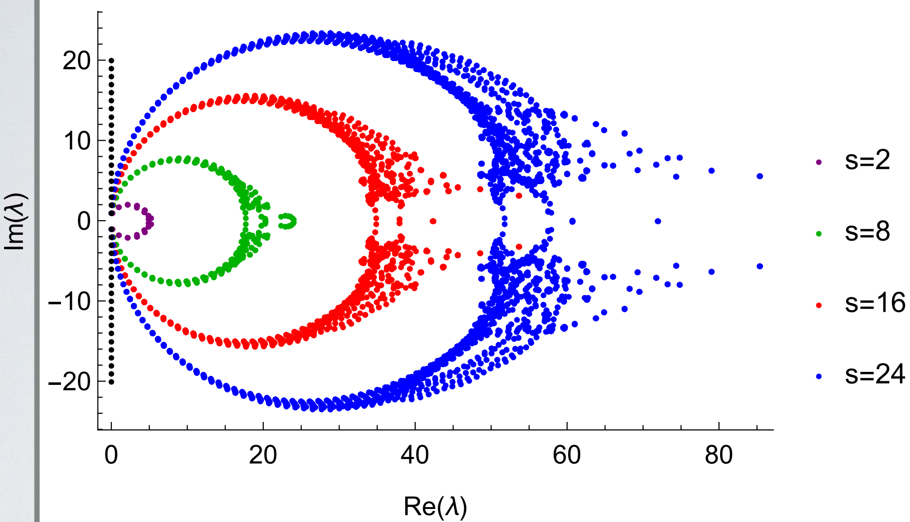


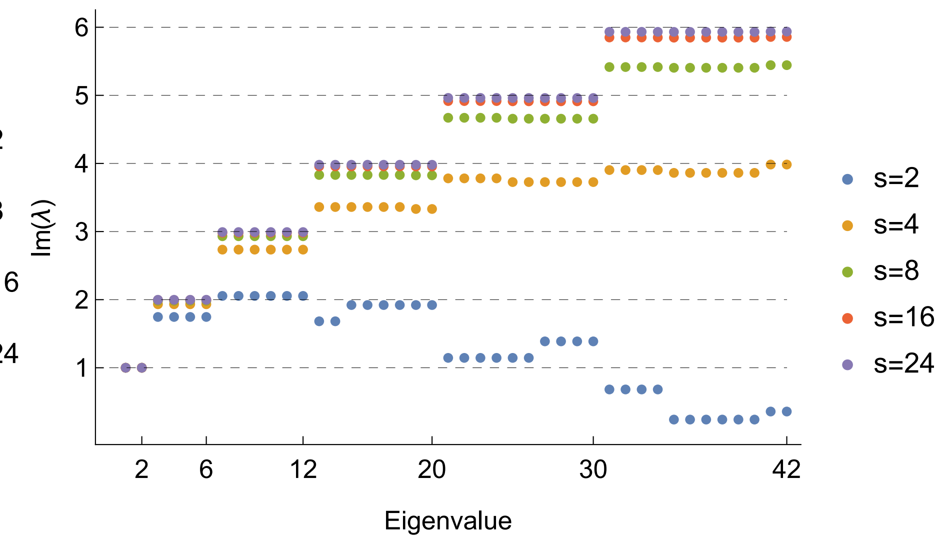
The spin connection is gauge field whose curl gives the local curvature or deficit angle

Geodesics and Parallel Transport is easy on a Sphere: In general use a Relaxation to fix Gauge Field

$$S^{(i)}_{\Delta} \equiv e^{i A^{\mu\nu}_{\Delta} \boldsymbol{R}_{\mu\nu}(i)} \quad \leftrightarrow \quad \Omega^{(i)}_{\Delta_{ijk}} \equiv \Omega_{ij}\Omega_{jk}\Omega_{ki}$$

2D DIRAC SPECTRA ON SPHERE





E = j + 1/2

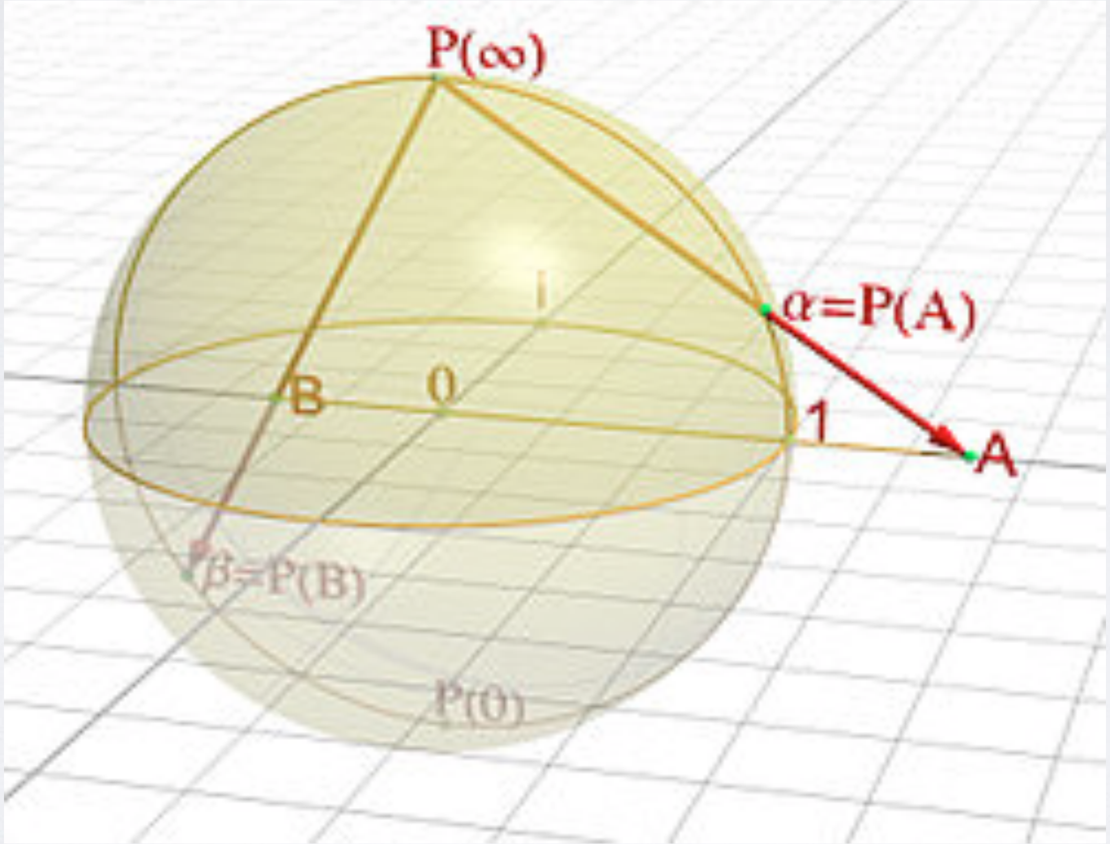
Exact is integer spacing for j = 1/2, 3/2, 5/2 ...

Exact degeneracy 2j + 1: No zero mode in chiral limit!.

QUANTUM COUNTER TERMS

TEST 2D ISING/PHI 4TH ON THE RIEMANN SPHERE

projection $$\xi = \tan(\theta/2)e^{-i\phi} = \frac{x+iy}{1+z}$$



Conformal Projection + Weyl Rescaling to the Sphere

EXACT SOLUTION TO C = 1/2 CFT

Exact Two point function

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \to \frac{1}{|1 - cos\theta_{12}|^{\Delta}}$$

$$\Delta = \eta/2 = 1/8 \qquad\qquad x^2 + y^2 + z^2 = 1$$

4 pt function $\quad (x_1, x_2, x_3, x_4) = (0, z, 1, \infty)$

$$g(0, z, 1, \infty) = \frac{1}{2|z|^{1/4}|1-z|^{1/4}}[|1 + \sqrt{1-z}| + |1 - \sqrt{1-z}|]$$

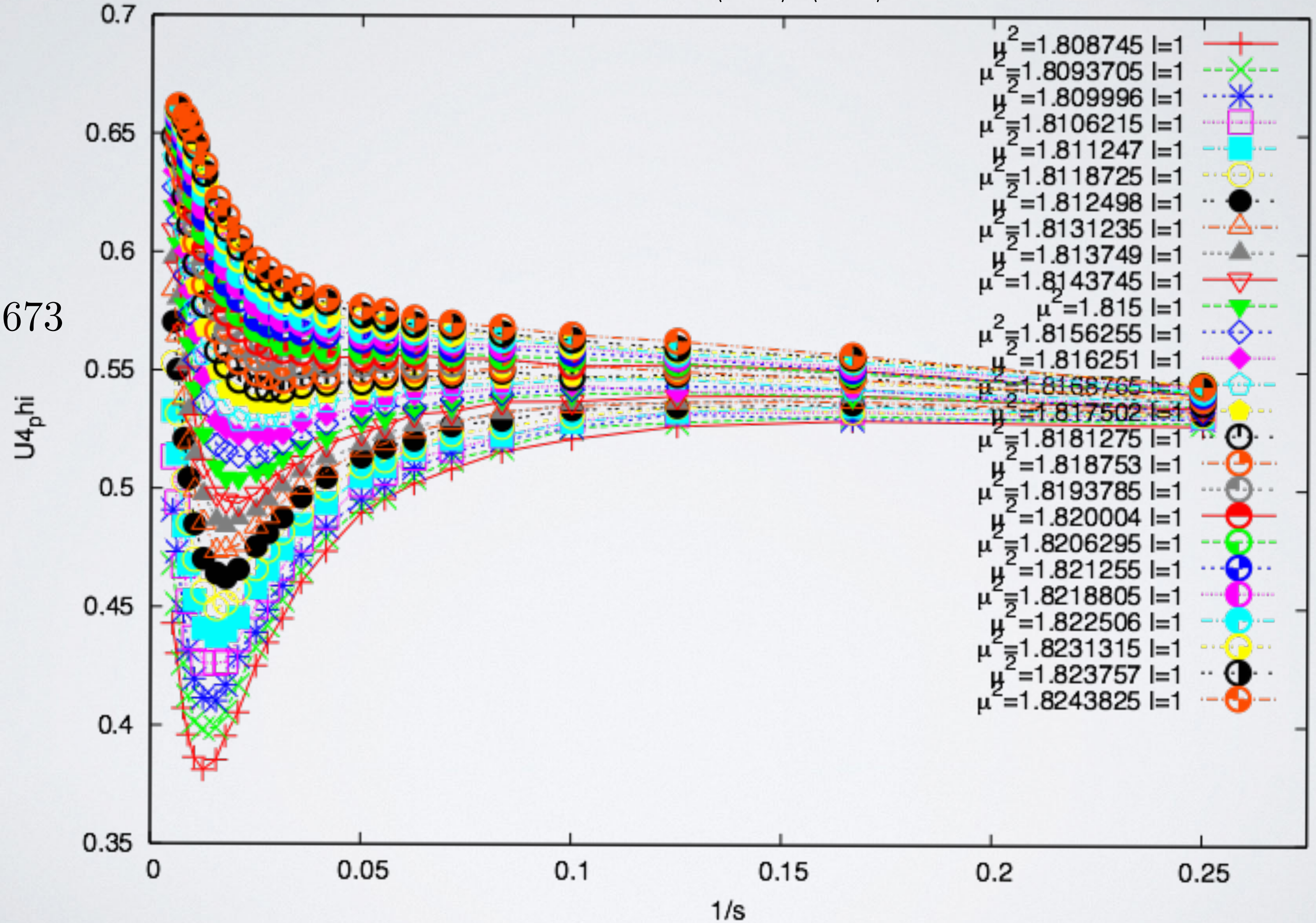
Critical Binder Cumulant $\quad U_B^* = 1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle\langle M^2\rangle} = 0.567336$

Dual to Free Fermion

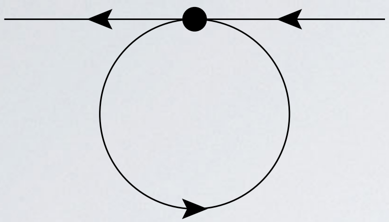
BINDER CUMULANT NEVER CONVERGES

$$U_B = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle \, \langle M^2 \rangle}$$

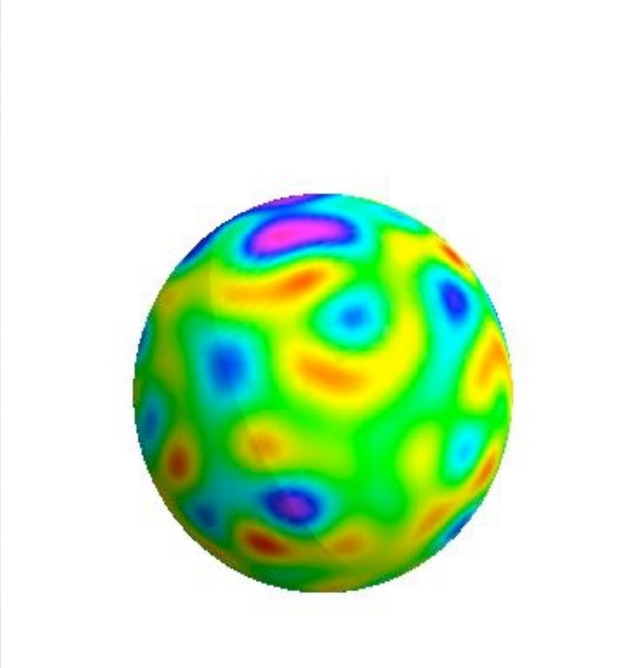
0.5673



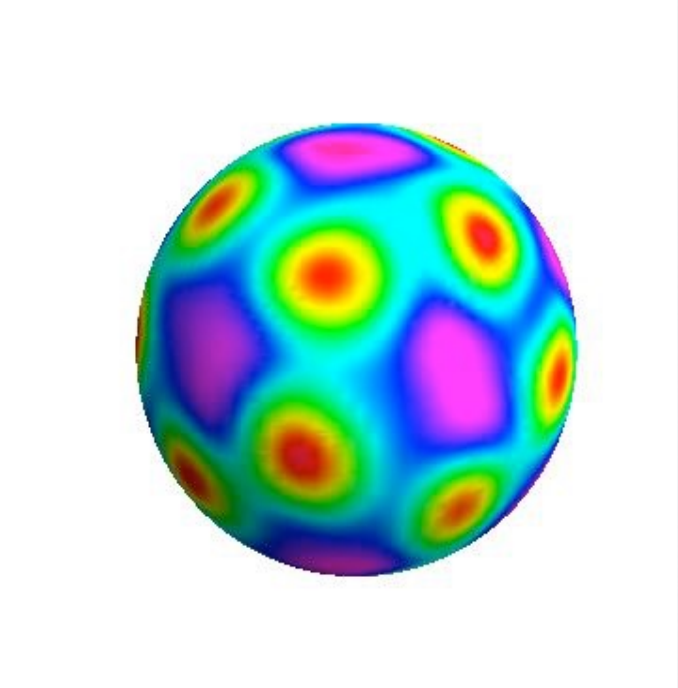
UV DIVERGENCE BREAKS ROTATIONS



$$\delta m^2 = \lambda \langle \phi(x)\phi(x) \rangle \to \frac{1}{K_{xx}}$$



one configuration

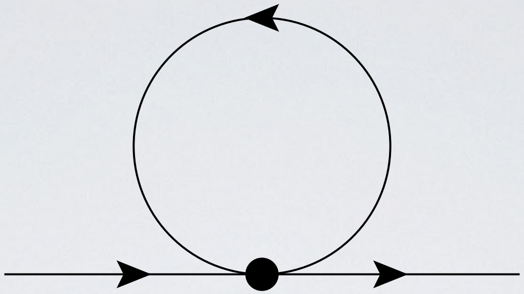


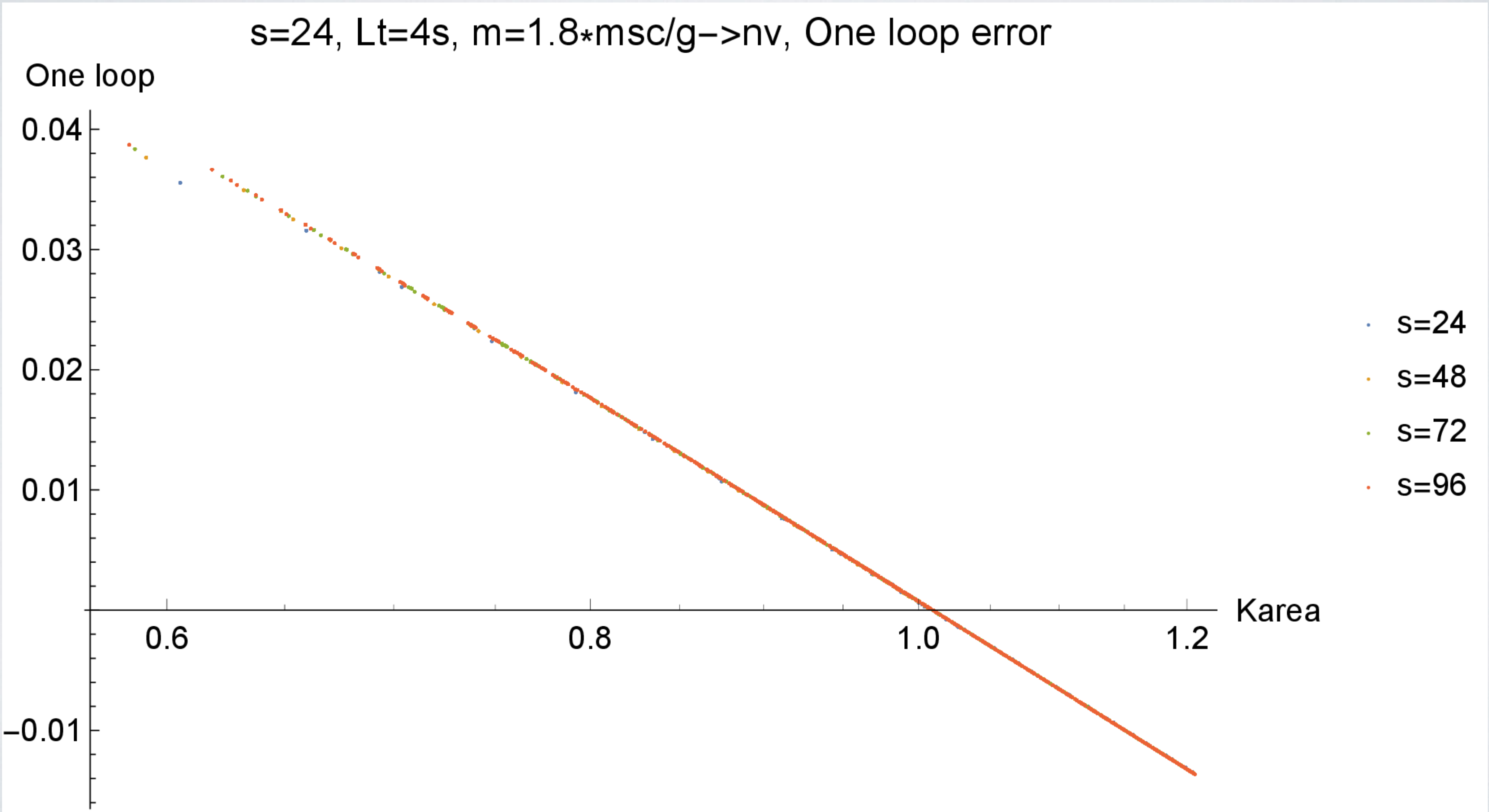
average of config.

ONE LOOP COUNTER TERM

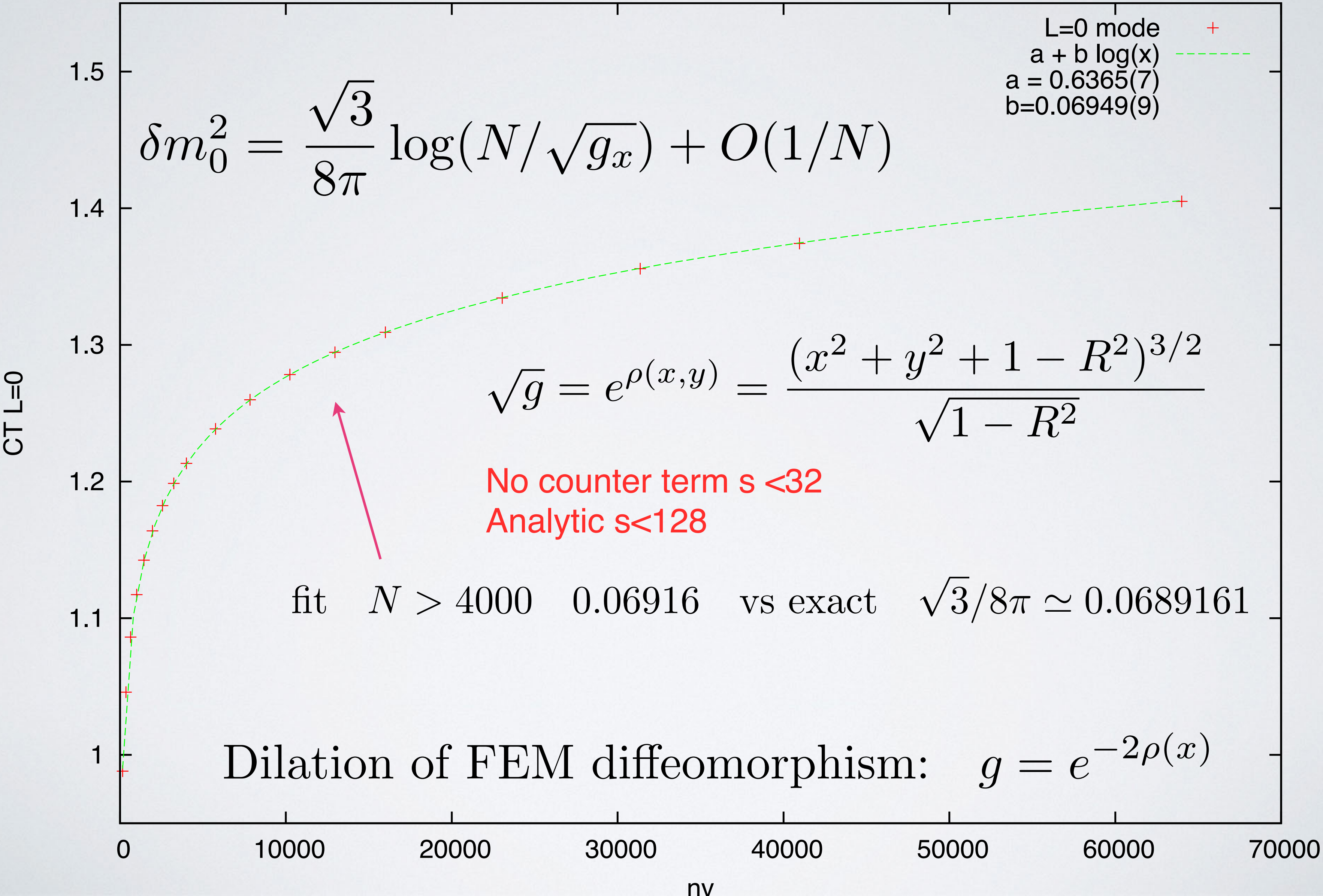
$$\Delta m_i^2 = 6\lambda \left[K^{-1}\right]_{ii} \simeq \frac{\sqrt{3}}{8\pi}\lambda \log(1/m_0^2 a_i^2) = \frac{\sqrt{3}}{8\pi}\lambda \log(N_s) + \frac{\sqrt{3}}{8\pi}\lambda \log(a^2/a_i^2)$$

$$\delta\mu_i^2 = -6\lambda(\left[K^{-1}\right]_{ii} - \frac{1}{N_s}\sum_{j=1}^{N_s}\left[K^{-1}\right]_{jj})$$




s=24, Lt=4s, m=1.8*msc/g->nv, One loop error
One loop
0.04
0.03
0.02
0.01
−0.01
0.6
0.8
1.0
1.2
Karea
s=24
s=48
s=72
s=96

MODEL OF COUNTER TERM



$$\delta m_0^2 = \frac{\sqrt{3}}{8\pi} \log(N/\sqrt{g_x}) + O(1/N)$$

$$\sqrt{g} = e^{\rho(x,y)} = \frac{(x^2+y^2+1-R^2)^{3/2}}{\sqrt{1-R^2}}$$

No counter term s <32
Analytic s<128

fit $\quad N > 4000 \quad 0.06916 \quad$ vs exact $\quad \sqrt{3}/8\pi \simeq 0.0689161$

Dilation of FEM diffeomorphism: $\quad g = e^{-2\rho(x)}$

NOW BINDER CUMULANT CONVERGES

$$U_{2n}(\mu^2, \lambda, s) = U_{2n,cr} + a_{2n}(\lambda)[\mu^2 - \mu^2_{cr}]s^{1/\nu} + b_{2n}(\lambda)s^{-\omega}$$

FIT:

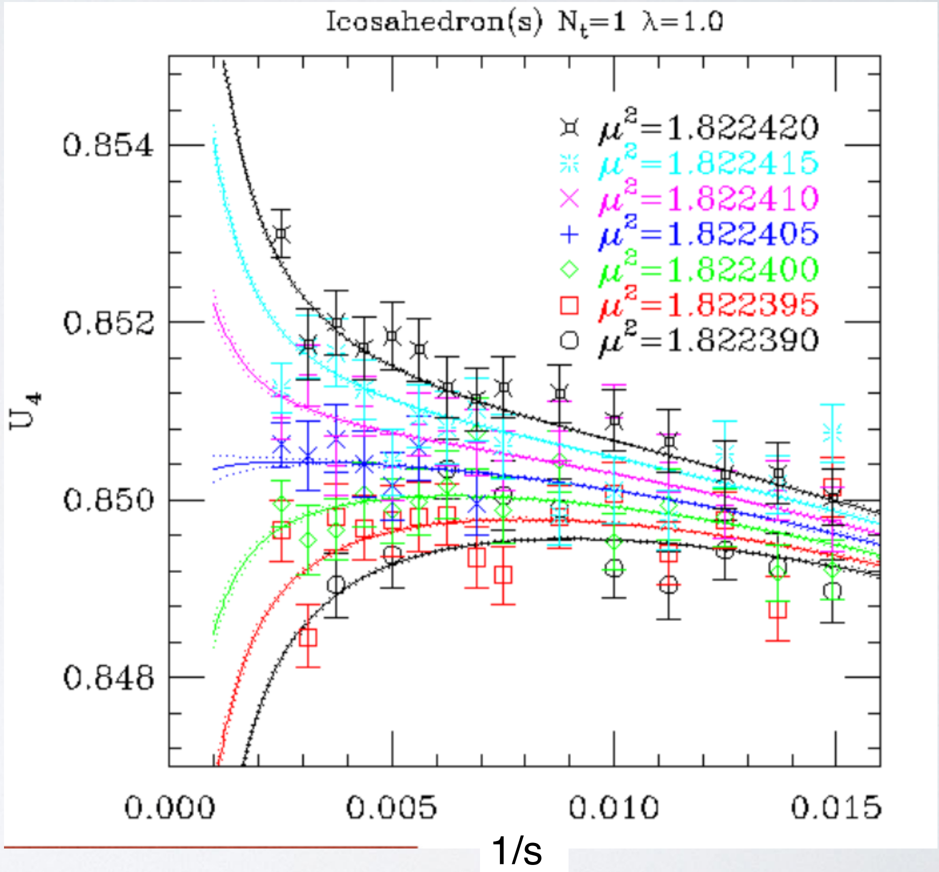
$$U_4 = 0.85081(10)$$

EXACT:

$$U_4^{exact} = 0.851021(5)$$

HIGHER MOMENT 2n = 4,6,8,10,12

$$U_6 = 0.77280(13)$$

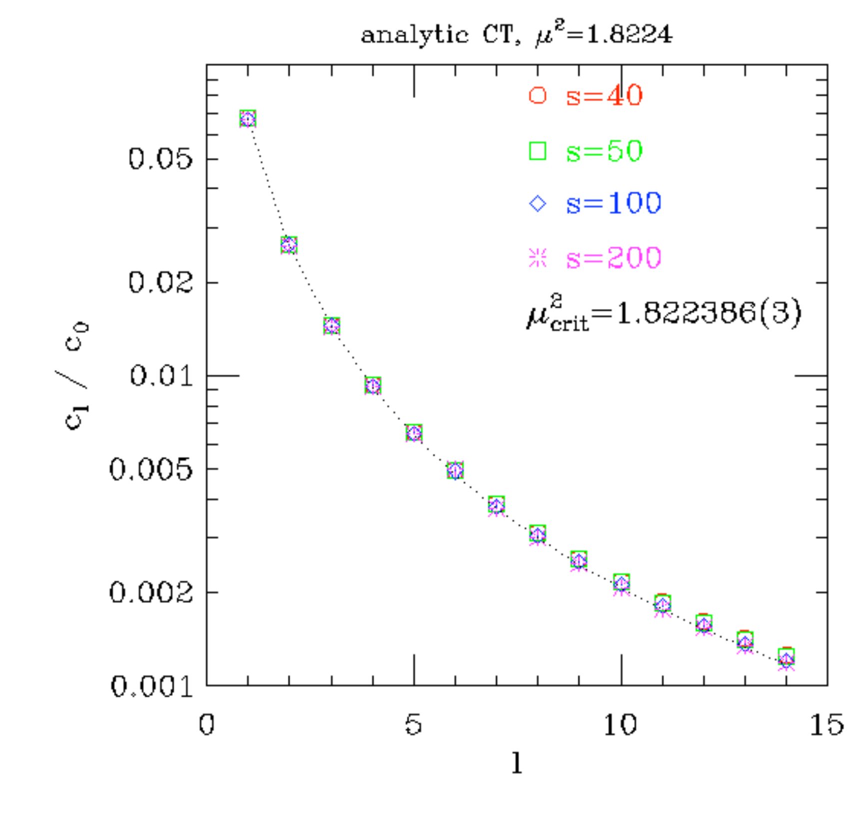


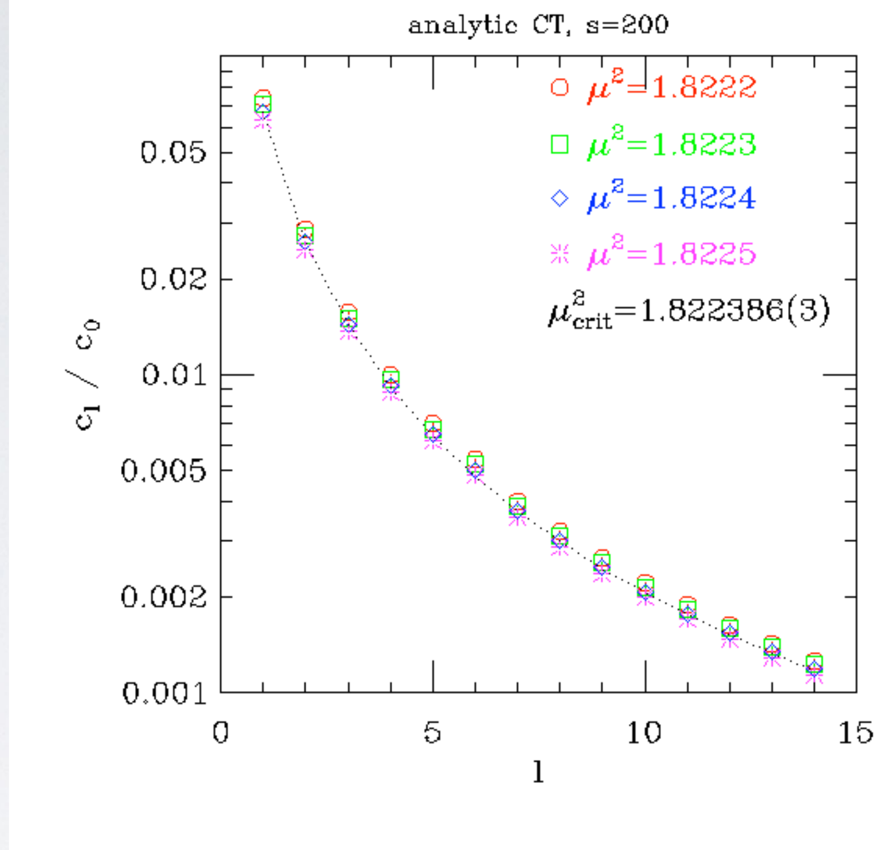
$$U_4 = \frac{3}{2}[1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle\,\langle M^2\rangle}]$$

Simultaneous fit for s up 800: E.G. 6,400,002 Sites on Sphere

$$\mu^2_{cr} = 1.82240070(34)$$

$$dof = 1701 \quad , \quad \chi^2/dof = 1.026$$





$$\int_{-1}^{1} dz \left(\frac{2}{1-z}\right)^{1/8} P_l(z) \qquad \Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

$$\Longrightarrow \frac{16}{7}, \frac{16}{35}, \frac{48}{161}, \frac{816}{3565}, \frac{12240}{64883}, \frac{493680}{3049501}, \frac{33456}{234577}, \frac{55760}{435643}, \frac{3602096}{30930653}, \frac{20129360}{187963199}, \frac{541373840}{5450932771} \cdots$$

Very fast cluster algorithm:
Brower,Tamayo 'Embedded Dynamics for phi 4th Theory" PRL 1989. Wolff single cluster + plus Improved Estimators etc

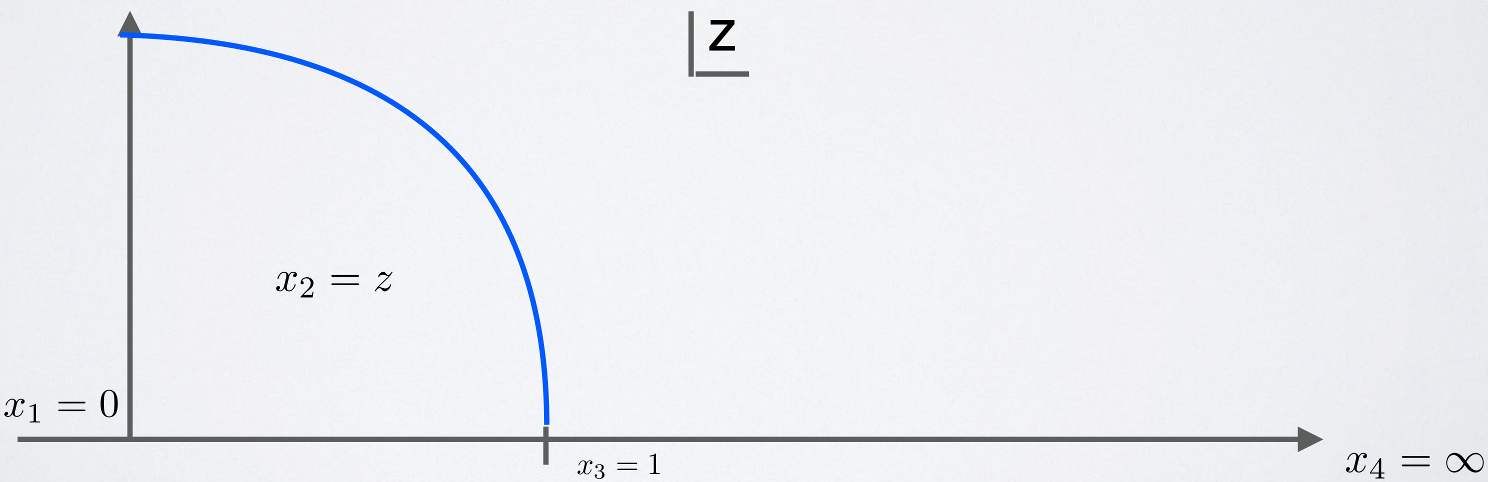
EXACT FOUR POINT FUNCTION

OPE Expansion: $\phi \times \phi = \mathbf{1} + \phi^2$ or $\sigma \times \sigma = \mathbf{1} + \epsilon$

$$g(u,v) = \frac{\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle}{\langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle}$$

$$= \frac{1}{\sqrt{2}|z|^{1/4}|1-z|^{1/4}} \left[|1+\sqrt{1-z}| + |1-\sqrt{1-z}| \right]$$

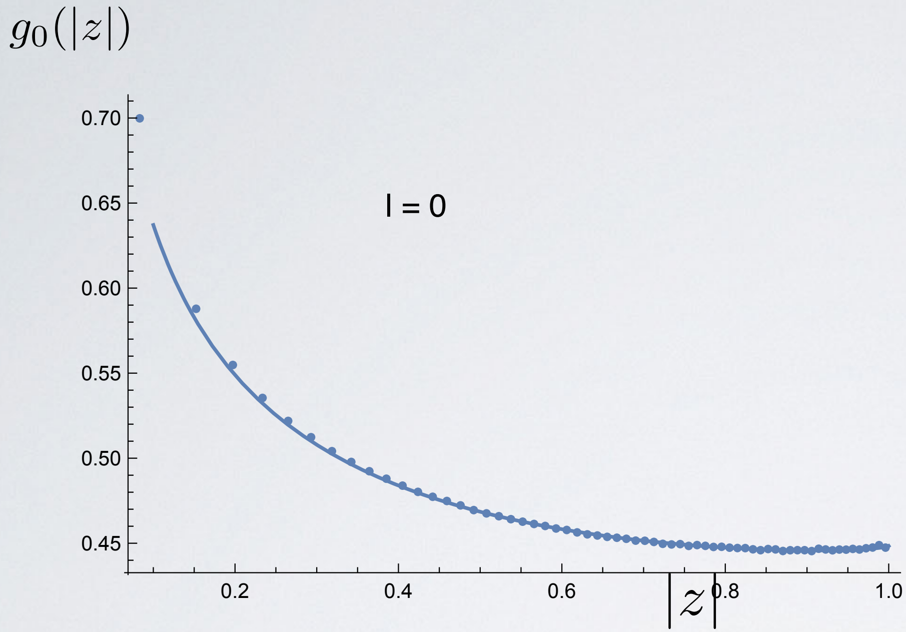
Crossing Sym: $|1+\sqrt{1-z}| + |1-\sqrt{1-z}| = \sqrt{2 + 2\sqrt{(1-z)(1-\bar{z})} + 2\sqrt{z\bar{z}}}$



$u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 x_{24}^2}$, $v = \frac{r_{14}^2 r_{32}^2}{r_{13}^2 r_{24}^2}$ where $r_{ij}^2 = (\vec{r}_i - \vec{r}_j)^2 = 2(1-\cos\theta_{ij})$

$u = z\bar{z}$, $v = (1-z)(1-\bar{z})$

2 TO 2 SCATTERING DATA

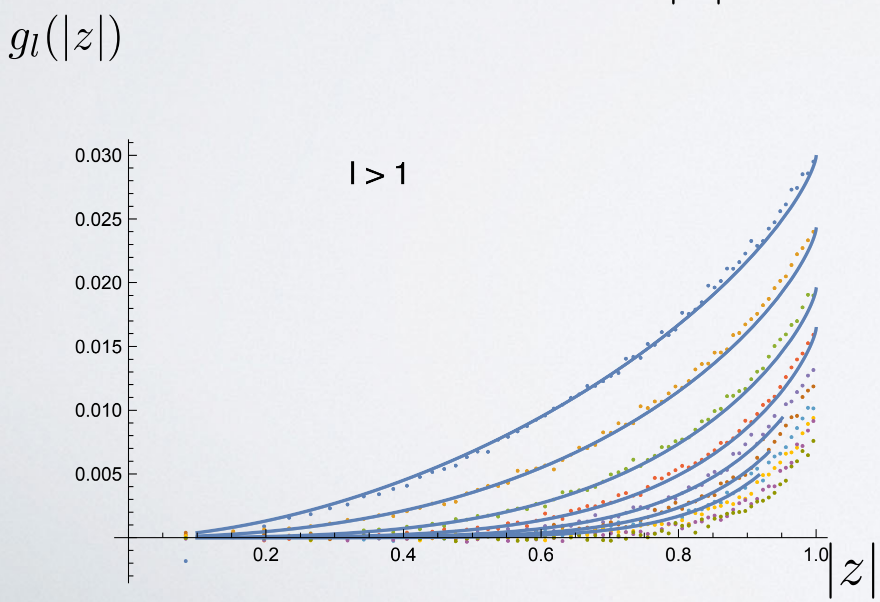


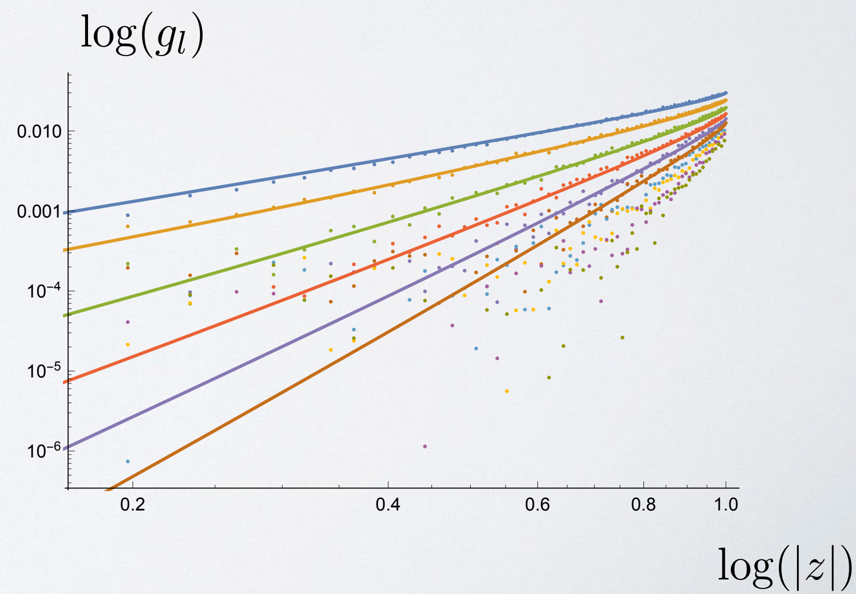
ZERO PARAMETER FIT

s= 10 Run for 1/2 hour

$$g(u,v) = \sum_l g_l(|z|)\cos(l\theta)$$

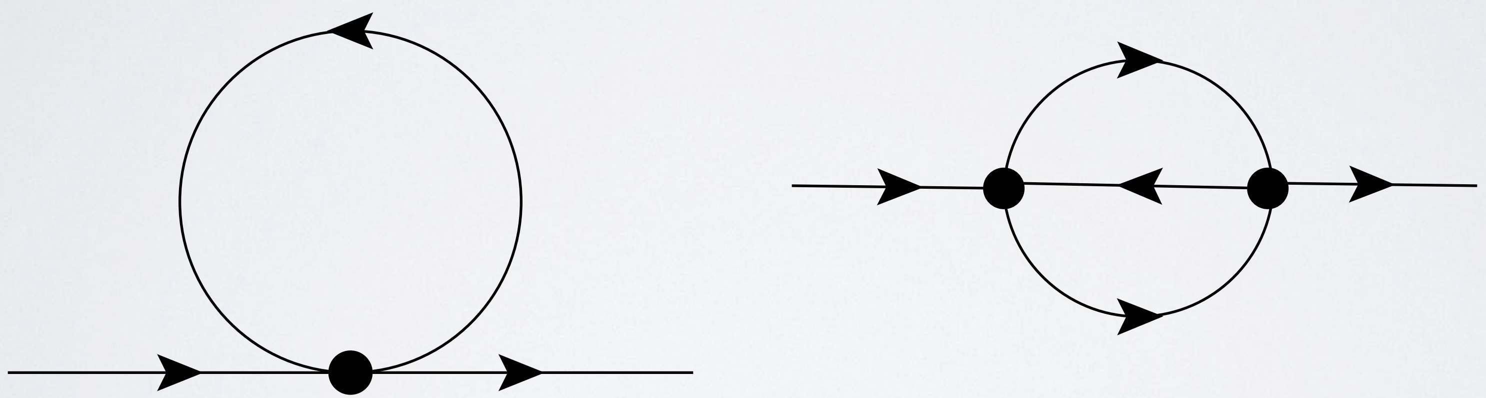
$$z = |z|e^{il\theta}$$

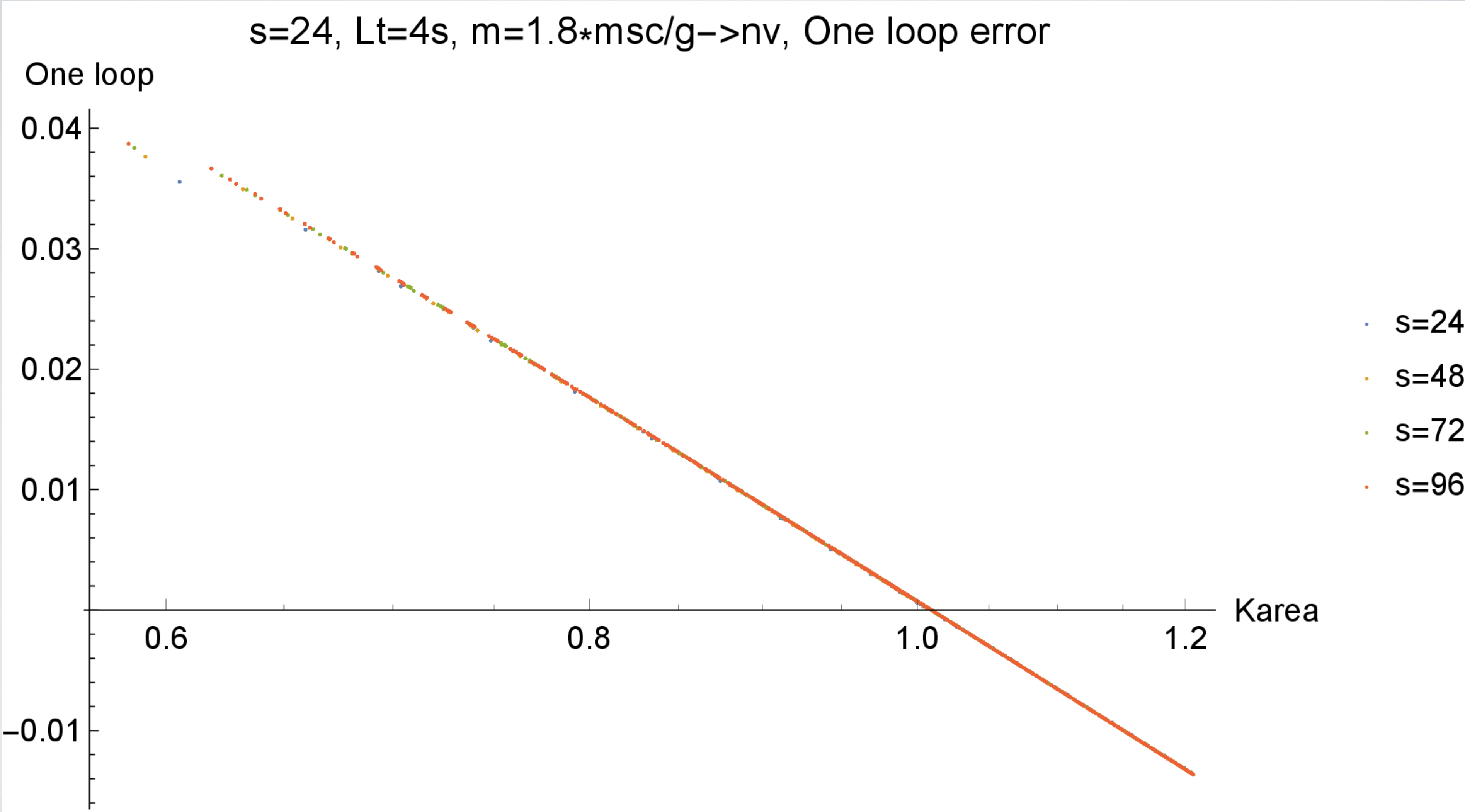


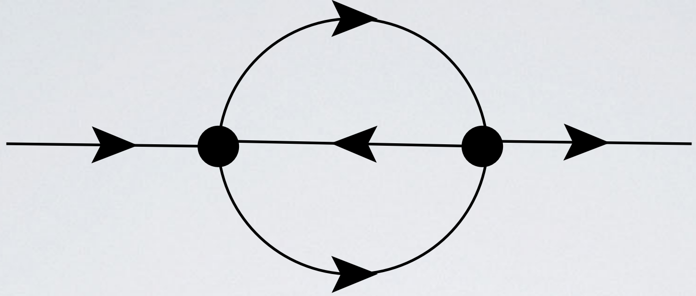


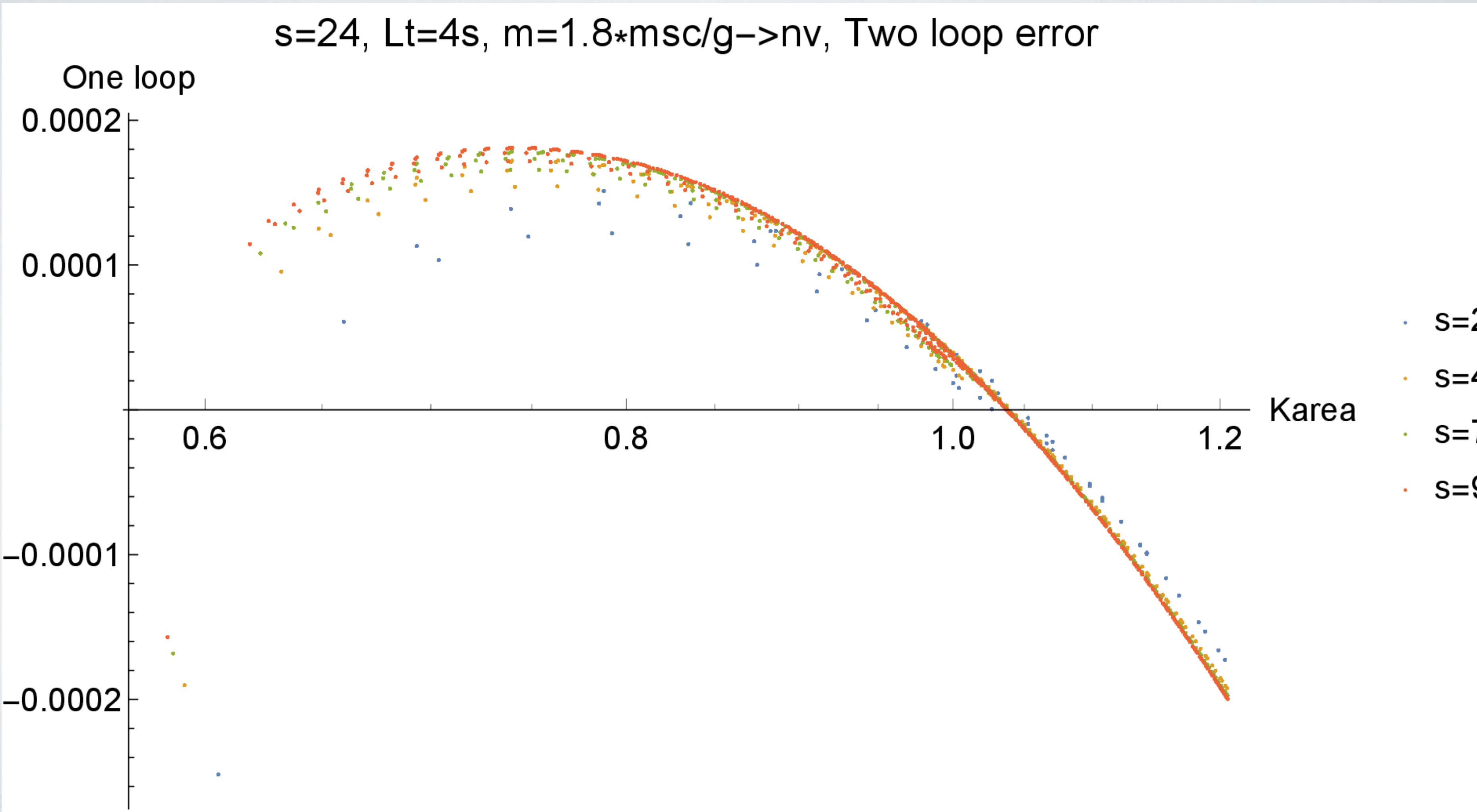
FUTURE

COUNTERTERM IN 3D




s=24, Lt=4s, m=1.8*msc/g->nv, One loop error
One loop
0.04
0.03
0.02
0.01
−0.01
0.6
0.8
1.0
1.2
Karea
s=24
s=48
s=72
s=96





s=24, Lt=4s, m=1.8*msc/g->nv, Two loop error
One loop
0.0002
0.0001
-0.0001
-0.0002
0.6
0.8
1.0
1.2
Karea

LESSON: QFE NEEDS COUNTER TERMS

APPROACHES

(i) Explicitly Subtract Finite Terms for Super Renormalized Theories

(ii) Pauli-Villars* 1949 (or Feynman and Stuekelberg)

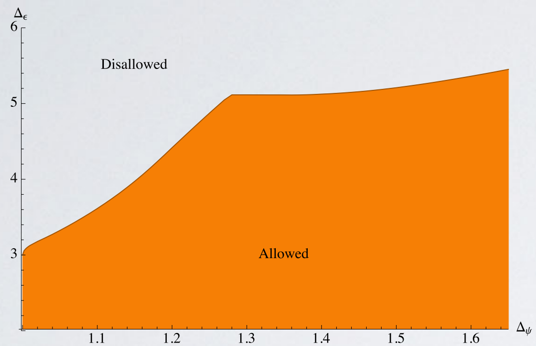
$$\frac{1}{p^2} - \frac{1}{p^2 + M_{PV}^2} \equiv \frac{1}{p^2 + p^4/M_{PV}^2} \qquad 1/\xi \ll M_{PV} \ll \pi/a$$

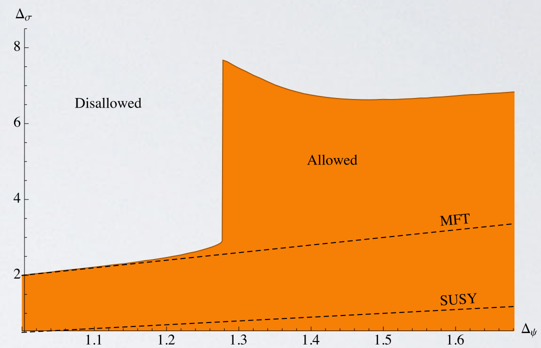
(iii) Working on smooth methods for D = 4 & Non-Abelian Gauge Theory

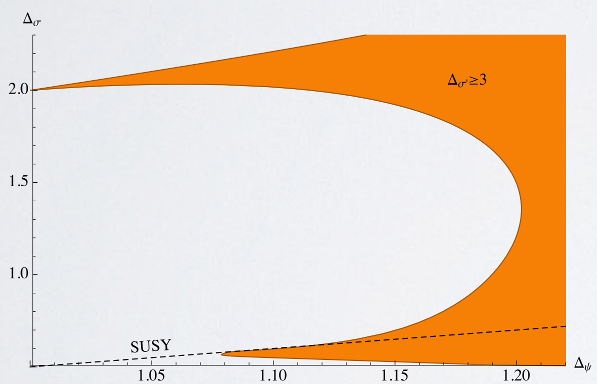
(iv) Would prefer not to have to use Quenched Randomize Simplicial Lattices but it may also work?

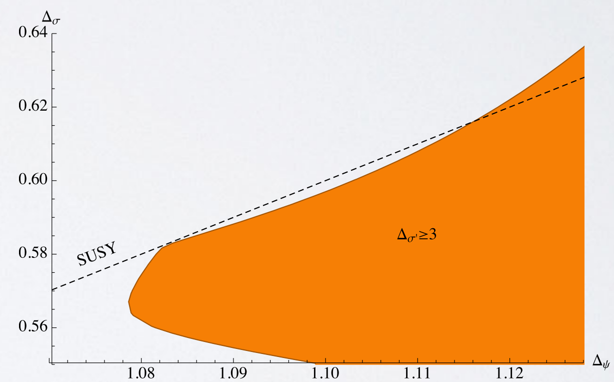
Bootstrapping 3D Fermions

Luca Iliesiu[a], Filip Kos[b], David Poland[b], 1508.00012v1.









$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{N}\bar{\psi}_i(\gamma^\mu\partial_\mu + g\phi)\psi_i - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4$$

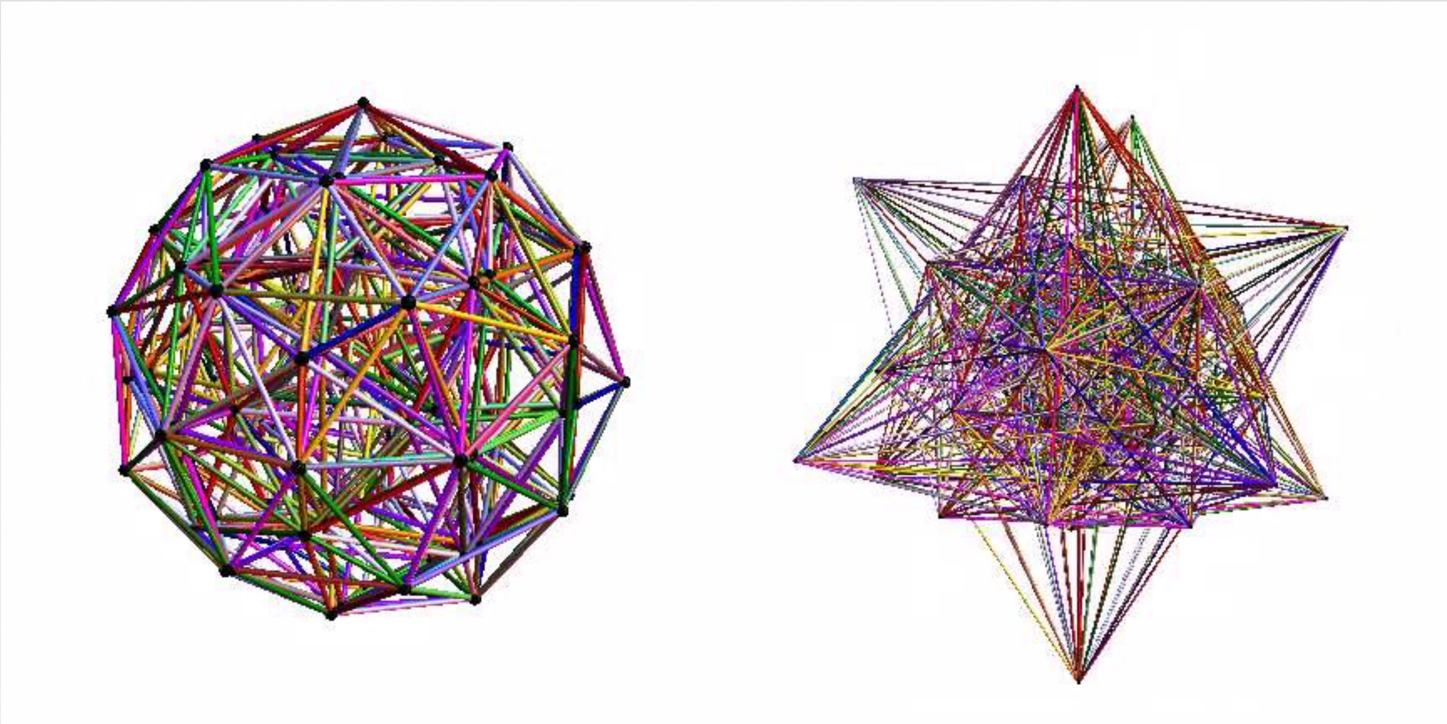
Gross Neuve (-Yukawa)

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{g}{2}\phi\bar{\psi}\psi - \frac{1}{8}\left(g\phi^2 + h\right)^2$$

N =1 SUSY-Ising

600 CELL ON S3

[HTTPS://EN.WIKIPEDIA.ORG/WIKI/600-CELL](https://en.wikipedia.org/wiki/600-cell)



16 vertices of the form:[3] (±½, ±½, ±½, ±½),

8 vertices obtained from (0, 0, 0, ±1) by permuting coordinates.

96 vertices are obtained by taking even permutations of ½ (±ϕ, ±1, ±1/ϕ, 0).

https://en.wikipedia.org/wiki/List_of_regular_polytopes_and_compounds#Five-dimensional_regular_polytopes_and_higher

QFE PLANS

- COMPUTATION:
  - 2+1 Radial Phi 4th/3D Ising CFT (with cluster algorithm)
  - Extend Peter Boyle's GRID to HMC on Simplicial Spheres (Interesting 3D Problem for Dirac/Scalar Theories.)
  - 3 Sphere starting with 600 cell: 4 Sphere ?
- THEORY:
  - Prove QFE for super renormalizable theories
  - Renormalization of 4d non-Abelian FT
  - Clarity DEC for Quantum FT

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

Using Binder Cumulants

$$U_4 = \frac{3}{2}\left(1 - \frac{m_4}{3\ m_2^2}\right) \qquad m_n = \langle \phi^n \rangle$$

$$U_6 = \frac{15}{8}\left(1 + \frac{m_6}{30\ m_2^3} - \frac{m_4}{2\ m_2^2}\right)$$

$$U_8 = \frac{315}{136}\left(1 - \frac{m_8}{630\ m_2^4} + \frac{2\ m_6}{45\ m_2^3} + \frac{m_4^2}{18\ m_2^4} - \frac{2\ m_4}{3\ m_2^2}\right)$$

$$U_{10} = \frac{2835}{992}\left(1 + \frac{m_{10}}{22680\ m_2^5} - \frac{m_8}{504\ m_2^4} - \frac{m_6\ m_4}{108 m_2^5} + \frac{m_6}{18\ m_2^3} + \frac{5\ m_4^2}{36\ m_2^4} - \frac{5\ m_4}{6\ m_2^2}\right)$$

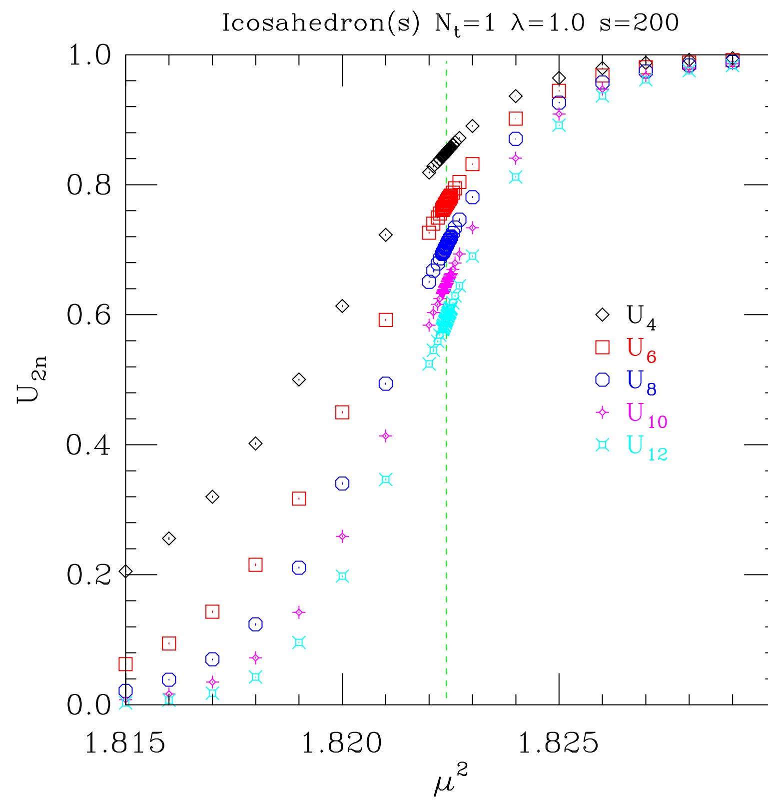
$$U_{12} = \frac{155925}{44224}\left(1 - \frac{m_{12}}{1247400\ m_2^6} + \frac{m_{10}}{18900\ m_2^5} + \frac{m_8\ m_4}{2520\ m_2^6} - \frac{m_8}{420\ m_2^4} + \frac{m_6^2}{2700 m_2^6} - \frac{m_6 m_4}{45\ m_2^5} + \frac{m_6}{15\ m_2^3} - \frac{m_4^3}{108\ m_2^6} + \frac{m_4^2}{4\ m_2^4} - \frac{m_4}{m_2^2}\right)$$

In infinite volume

U_{2n}=0 in disordered phase

U_{2n}=1 in ordered phase

0<U_{2n}<1 on critical surface



- $U_{2n,cr}$ are universal quantities.
- Deng and Blöte (2003): $U_{4,cr}$=0.851001
- Higher critical cumulants computable using conformal 2n-point functions: Luther and Peschel (1975) Dotsenko and Fateev (1984)

LINEAR FINITE ELEMENT APPROACH

$$S = \frac{1}{2}\int_{\mathcal{M}} d^D x \sqrt{g}\, [g^{\mu\nu}\partial_\mu \phi(x) \partial_\nu \phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)]$$

$$\Downarrow$$

$$\begin{aligned} I_\sigma &= \frac{1}{2}\int_\sigma d^D y [\vec{\nabla}\phi(y)\cdot\vec{\nabla}\phi(y) + m^2\phi^2(y) + \lambda\phi^4(y)] \\ &= \frac{1}{2}\int_\sigma d^D \xi \sqrt{g}\, [g^{ij}\partial_i\phi(\xi)\partial_j\phi^2(\xi) + m^2\phi^2(\xi) + \lambda\phi^4(\xi)] \end{aligned}$$

$$\Downarrow$$

$$I_\sigma \simeq \sqrt{g_0}\left[g_0^{ij}\frac{(\phi_i - \phi_0)(\phi_j - \phi_0)}{l_{i0}\, l_{j0}} + m^2\phi_0^2 + \lambda\phi_0^4\right]$$